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EDINBURGH MATHEMATICAL SOCIETY.

FIFTEENTH SESSION, 1896-97.

First Meeting, November 20th, 1896.

WM. PEDDIE, D.Sc., F.R.S.E., President, in the Chair.

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A. G. WALLACE, M.A.

[*The following Paper was read at Third and Fourth Meetings, 8th January and 12th February* 1897.]

On the Geometrical Representation of Elliptic Integrals of the First Kind.

By ALEX. MORGAN, M.A., B.Sc.

I.

When an expression has to be integrated which contains the square root of a rational function of the first or second degree, the integral can be expressed in terms of the ordinary algebraic functions or the elementary transcendental functions, viz., exponential and circular. But when the polynomial under the radical is higher than the second degree its integral in general can only be expressed by means of transcendentals of a higher kind. The particular case in which the expression under the square root is a cubic or quartic gives rise to a class of definite integrals called *Elliptic Integrals*, because by means of them, as we shall see, we can express the length of the arc of an ellipse or other central conic.

Legendre * considered the general elliptic integral $\int \frac{\mathrm{P}dx}{\sqrt{\mathrm{X}}}$,

where P is any rational function whatever of x, and X is a positive rational integral quartic function of x with real coefficients, and he showed that by partial integrations and by transformations this integral could be resolved into an algebraic part together with transcendentals always of three types †, viz.,

$$\int \frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}}, \int \frac{x^2dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}}, \int \frac{dx}{(1+nx^2)\sqrt{1-x^2 \,.\, 1-k^2x^2}},$$

where k is $+^{ve}$ and less than 1, and n is real or imaginary.

Without altering the type of the second integral we may write it

$$\int \frac{(1-k^2x^2)dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}} \quad \text{or} \quad \int \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$$

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* *Traité de Fonctions Elliptiques*, t. I., Chap. iii., iv. and v.

† *cf.* CAYLEY'S *Elliptic Functions*, § 1.

Putting $x = \sin\phi$, we thus have three kinds of elliptic integrals:

First kind, $\quad F(k, \phi) \quad = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}$

Second kind, $\quad E(k, \phi) \quad = \int_0^\phi \sqrt{1 - k^2\sin^2\phi}\, d\phi$

Third kind, $\quad \Pi(n, k, \phi) = \int_0^\phi \frac{d\phi}{(1 + n\sin^2\phi)\sqrt{1 - k^2\sin^2\phi}}.$

ϕ is called the *amplitude*, and is a real angle, k the *modulus*, and n the *parameter*.

The arc of an ellipse can be represented by an elliptic integral of the second kind. Thus in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

if we put
$$x = a\sin\phi,\ y = b\cos\phi$$
then
$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= (a^2\cos^2\phi + b^2\sin^2\phi)d\phi^2 \\ &= [a^2 - (a^2 - b^2)\sin^2\phi]d\phi^2 \\ &= a^2(1 - e^2\sin^2\phi)d\phi^2 \\ \therefore\ s &= a\int \sqrt{1 - e^2\sin^2\phi}\,d\phi \\ &= aE(e, \phi), \end{aligned}$$

the arc being measured from the extremity of the minor axis.

It is thus very easy to find a curve whose arc will represent an elliptic integral of the second kind, but it has always been a difficult problem to give a complete geometrical representation of integrals of the first kind.*

Historical Note.

The name *elliptic functions* is somewhat of a misnomer, as the whole theory of these functions is based on the first elliptic integral $F(k, \phi)$ which can *not* be represented by an arc of an ellipse.

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* For references to the numerous attempts to solve the problem, see Note V. in the Appendix of Müller's edition of Enneper's *Elliptische Functionen* (Halle, 1890).

It was LEGENDRE (*Mémoire sur les transcendantes elliptiques*, 1793 : *Exercices du Calcul Intégral sur divers ordres de Transcendantes et sur les Quadratures*, 3 vols. 1811–1819 ; and *Traité des Fonctions Elliptiques*, 3 vols. 1825–32) who discovered most of the important properties of the new functions, and invented a notation for them ; but the founder of the modern theory of elliptic functions may fairly enough be said to be ABEL (his earlier Memoirs on the subject appeared in *Crelle's Journal*, 1826–9. They are collected in *Œuvres Complètes de N. H. Abel par B. Holmboe*, 1839. His great Memoir on Transcendental Functions is published by the French Academy in *Mémoires des Savants Etrangers*, t. vii., 1841. The most recent edition of his works is *N. H. Abel. Tableau de sa vie et de son action scientifique. Par C. A. Bjerknes*, 1885).

About 1823 ABEL pointed out that in

$$F(k, \phi)=\int_0^{\phi}(1-k^2\sin^2\phi)^{-\frac{1}{2}}d\phi$$

$F(k, \phi)$ is of the nature of an inverse function, and that if we put $u=F(k, \phi)$ then we should study the properties of the amplitude ϕ as a function of u, and not u as a function of ϕ. Legendre laid great stress on the elliptic integrals, and tried to deduce the properties of elliptic functions from them, but Abel pointed out that by following Legendre's method mathematicians were making the same mistake as if they had tried to deduce the theorems of trigonometry by studying the properties of the *inverse* circular functions as deduced from the circular integrals. It is Abel's idea of the *inversion* of the first elliptic integral, and his discovery of the double periodicity of elliptic functions that have led to the wonderful recent advances in the theory of elliptic and higher transcendentals in Germany and France.

It is interesting to trace the genesis of elliptic functions out of the early attempts of geometers to rectify the ellipse.

MACLAURIN (*A Treatise of Fluxions*, Edinburgh 1742) and D'ALEMBERT (*Des Différentielles qui se rapportent à la rectification de l'ellipse ou de l'hyperbole* in the *Histoire de l'Acad. de Berlin*, 1746) seem to have been the first to study integrals which could be expressed by the arcs of an ellipse or hyperbola. They found a great many such integrals, but their results were disjointed.

Next, FAGNANO (*Produzioni Matematiche Del Marchese Giulio Carlo De' Toschi Di Fagnano.* 2 vols. Pesaro 1750) proved that in any given ellipse or hyperbola we can in an infinite number of ways find two arcs of which the difference is expressible by an algebraic quantity.

It was EULER, however, who first tried to develop a general theory out of those scattered theorems. He clearly foresaw that with a suitable notation a new kind of functions, founded on the properties of the arcs of ellipses, would arise which would become as general and as important in the higher analysis as were the circular and logarithmic functions. At vol. X. p. 4 of *Novi Commentarii Acad. Sc. Petropoli* (1764), Euler says "Imprimis autem hic idoneus signandi modus desiderari videtur, cujus ope arcus elliptici aeque commode in calculo exprimi queant, ac jam logarithmi et arcus circulares ad insigne Analyseos per idonea signa in calculum sint introducti. Talia signa novam quandam calculi speciem suppeditabunt. . . . . ." We shall see

that this passage had afterwards a most important effect on the work of Legendre.

Also Euler (*Novi Com.*, Vols. vi. and vii., 1761) discovered the method of integrating by algebraical functions the differential equation now known as *Euler's Equation*, viz.,

$$X^{-\frac{1}{2}}dx + Y^{-\frac{1}{2}}dy = 0$$

where X is a quartic function of x and Y is the same function of y, say

$$X = ax^4 + bx^3 + cx^2 + ex + f$$
$$Y = ay^4 + by^3 + cy^2 + ey + f.$$

The integral is $$\left(\frac{X^{\frac{1}{2}} - Y^{\frac{1}{2}}}{x-y}\right)^2 = C + a(x+y)^2 + b(x+y).$$

He thus found the complete algebraical integral of a differential equation composed of two similar terms, *whose integrals taken separately are not algebraic but only expressible by the arcs of an ellipse or other conic section.*

John Landen (*Philosophical Transactions*, 1775; *Mathematical Memoirs*, 1780) proved that every arc of a hyperbola can be expressed in terms of the arcs of two ellipses. This theorem was an important step in the simplification of the theory of such arcs.

Legendre in 1786 published the first of his investigations in connection with the subject. In that year there appeared his *Mémoire sur les intégrations par d'arcs d'ellipse* in *Mém. de l'Acad. des Sciences de Paris.* Among other things he proved that in an infinite series of ellipses formed according to the same law the rectification of one of the ellipses can be reduced to that of two others chosen at will from the series.

Not, however, until his attention was arrested by Euler's discovery of 1761 and prediction of 1764 did Legendre perceive the way in which the new functions were to be attacked with success. He was led by Euler's equation to examine all the transcendentals contained in $\int \frac{Pdx}{\sqrt{X}}$ where P is any rational function of x, and X a rational function of x of the fourth degree. He classified the integrals included under this general form into three kinds, and developed a notation and theory for the reduced integrals as had been desiderated by Euler in 1764. These results are contained in his *Mémoire sur les Transcendantes elliptiques*, 1793. All the investigations of Legendre were afterwards collected and published in his *Exercices du Calcul Intégral* etc., and his *Traité des Fonctions Elliptiques* already mentioned.

Jacobi (*Fundamenta Nova Theoriæ Functionum Ellipticarum*, 1829, and Memoirs in *Crelle's Journal*, 1828–1858) carried out Abel's idea of the inversion of the first elliptic integral and introduced a notation to take the place of Legendre's. He took $F(k, \phi)$ as the independent variable and put u for it, calling ϕ the amplitude of u or shortly $\phi = \operatorname{am} u$. Then $\sin\phi$, $\cos\phi$ and $\Delta\phi\left(=\sqrt{1-k^2\sin^2\phi}\right)$ were the sine, cosine, and Δ of the amplitude of u, or as he wrote them $\sin\operatorname{am} u$, $\cos\operatorname{am} u$, $\Delta\operatorname{am} u$. In the *Fundamenta Nova* he developed with great elegance the properties of these three elliptic functions

of u. He also changed the meanings of Legendre's symbols E and Π for the elliptic integrals of the second and third kinds into those denoted by the equations

$$\mathrm{E}u = \int_0^u \mathrm{dn}\, u\, du, \qquad \Pi(u, a) = \int_0^u \frac{k^2 \,\mathrm{sn}\, a\, \mathrm{cn}\, a\, \mathrm{dn}\, a\, \mathrm{sn}^2 u}{1 - k^2 \,\mathrm{sn}^2 u\, \mathrm{sn}^2 a} du$$

He introduced new functions zeta, theta, and eta defined as follows :—

$$\mathrm{Z}u = \mathrm{E}u - \frac{u\mathrm{E}}{\mathrm{K}}, \qquad \log\frac{\Theta u}{\Theta o} = \int_0^u \mathrm{Z}u\, du, \text{ or } \Theta u = \Theta 0 \exp. \int_0^u \mathrm{Z}u\, du$$

$$\mathrm{H}u = \sqrt{k}.\Theta u.\mathrm{sn}\, u.$$

It was GUDERMANN (*Theorie der Modular Functionen* in *Crelle's Journal*, vol. xviii. p. 12) who proposed the abbreviations sn, cn, dn for sin am, cos am, Δ am, and Dr GLAISHER (*Messenger of Math.*, vol. xi. p. 86) who introduced the notation

	$\mathrm{ns}\, u$	$\mathrm{nc}\, u$	$\mathrm{nd}\, u$	$\mathrm{sc}\, u$	$\mathrm{cd}\, u$ etc.
for	$\frac{1}{\mathrm{sn}\, u}$	$\frac{1}{\mathrm{cn}\, u}$	$\frac{1}{\mathrm{dn}\, u}$	$\frac{\mathrm{sn}\, u}{\mathrm{cn}\, u}$	$\frac{\mathrm{cn}\, u}{\mathrm{dn}\, u}$ etc.

In recent times no worker has done so much to develop the theory as WEIERSTRASS. While Jacobi's functions sn u, cn u, dn u are recognised as valuable for numerical work, it is granted on all hands that Weierstrass's functions

$$\sigma u = u\Pi_w\left(1 - \frac{u}{w}\right)e^{\frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2}}$$

($w = 2m\omega + 2m'\omega'$, where ω and ω' are the two periods of the function, and $m, m' = 0, \pm 1, \pm 2, \ldots \pm\infty$ but w cannot, as we see, be $=0$, therefore m and m' cannot be simultaneously $=0$)

and

$$pu = -\frac{d^2}{du^2}\log\sigma u$$

form the proper basis for the theory of elliptic functions. σu is the one from which Weierstrass evolves the theory of elliptic functions, but, from the point of view of the elliptic integral, pu is the one which, as we shall see, most naturally presents itself.

II.

Legendre (*Traité des Fonctions Elliptiques*, Vol. I. p. 35) showed that the integral $\mathrm{F}(k, \phi)$ was represented by the lemniscate

$$(x^2 + y^2)^2 - a^2(x^2 - y^2) = 0$$

in the particular case in which the modulus $k = \frac{1}{\sqrt{2}}$.

The equation of the lemniscate is satisfied by the values

$$x = a\cos\phi\sqrt{1-\tfrac{1}{2}\sin^2\phi}$$

$$y = \frac{a}{\sqrt{2}}\sin\phi\cos\phi$$

But $$ds^2 = dx^2 + dy^2$$

$$= \frac{a^2\,d\phi^2}{2(1-\tfrac{1}{2}\sin^2\phi)}$$

$$\therefore\ s = \frac{a}{\sqrt{2}}\int\frac{d\phi}{\sqrt{1-\tfrac{1}{2}\sin^2\phi}} = \frac{a}{\sqrt{2}}\mathrm{F}\left(\frac{1}{\sqrt{2}},\ \phi\right),$$

the arc s being measured from the point $\phi = 0$, *i.e.*, from the extremity $(a, 0)$ of the axis of the lemniscate.

An important result follows from this, viz., from the addition, subtraction, multiplication, and division of elliptic functions it follows that *arcs of a lemniscate can be added, subtracted, multiplied and divided algebraically just as the arcs of a circle can* *.

After much time spent on the problem, Legendre (*Traité des Fonctions Elliptiques*, Vol. I. p. 36) invented a sextic curve which represented, *for all values of* k, the function $\mathrm{F}(k, \phi)$ with an algebraic function subtracted.

He took the curve whose coordinates satisfy the equations

$$(1)\qquad \begin{cases} x = h\sin\phi(1+\tfrac{1}{3}m\sin^2\phi)\ \dagger \\ y = k'h\cos\phi(1+m-\tfrac{1}{3}m\cos^2\phi) \end{cases}$$

where k' is the complementary modulus, *i.e.* $= \sqrt{1-k^2}$.

But $ds = \sqrt{dx^2+dy^2} = h\sqrt{1-k^2\sin^2\phi}(1+m\sin^2\phi)d\phi$

$$\therefore\ (2)\ s = h\mathrm{E}(k,\ \phi) + \frac{mh}{3k^2}[(2k^2-1)\mathrm{E}(k,\ \phi) + k'^2\mathrm{F}(k,\ \phi)] - k^2\sin\phi\cos\phi\sqrt{1-k^2\sin^2\phi}$$

---

* As early as 1716, long before Legendre had discovered the method of multiplying and dividing elliptic functions, Fagnano was able to multiply and divide arcs of lemniscates (see *Methodo per misurare la Lemniscata*, pp. 343–368 of vol. 2 of his *Produzioni Matematiche*).

† Cayley at § 62 of his *Elliptic Functions* states the value of x and y erroneously.

In order that the second elliptic integral may disappear we must have

$$m = \frac{3k^2}{1 - 2k^2}, \text{ and therefore } h = \frac{3k^2}{k'^2 m} = \frac{1 - 2k^2}{k'^2}.$$

Hence we have

$$s = F(k, \phi) - \frac{k^2}{k'^2} \sin\phi \cos\phi \sqrt{1 - k^2 \sin^2\phi}.$$

The objection to this solution is that the arc s represents not $F(k, \phi)$ but this integral *minus an algebraic function.* This algebraic quantity can be made to vanish by a suitable choice of the ends of the arc *, but it is not in general zero, and hence the sextic curve invented is not a perfect representation of the first elliptic transcendental. †

The curve under consideration is, however, an interesting one. From (2) we see that k^2 may have any value provided it is not greater than $\frac{1}{2}$. On eliminating ϕ in (1) after having inserted the above values of h and m, we find the equation of the curve to be

$$y^2 = k^2(2 + x^{\frac{2}{3}})^2(1 - x^{\frac{2}{3}}) \tag{3}$$

Hence the curve is of the sixth degree, and since there are no odd powers of x and y the curve is divided into four equal and similar parts by the axes of coordinates. The curve is of the form given in Fig. 11. On inserting the values of h and m, (1) gives

$$x = \frac{\sin\phi}{k'^2}(1 - 2k^2 + k^2 \sin^2\phi)$$

$$y = \frac{\cos\phi}{k'}(1 + k^2 \sin^2\phi).$$

Hence, if $\phi = 0$ then $x = 0$, $y = \frac{1}{k'}$

if $\phi = \frac{\pi}{2}$ then $x = 1$, $y = 0$

---

* For example, if we integrate between the limits $\phi = 0$ and $\phi = \frac{\pi}{2}$ then the additional algebraic function obviously vanishes, and we get $s_1 = F\left(k, \frac{\pi}{2}\right)$ where s_1 is the fourth part of the curve.

† Yet Legendre remarks (F. E., vol. ii., p. 591) "Le problème . . . . de trouver une courbe algébrique dont les arcs représentent généralement la fonction elliptique de première espèce $F(k, \phi)$ *paraît n' admettre aucune autre solution.*" We shall see that we have travelled far since then.

that is, CA is the semimajor axis of the curve and its length is $\frac{1}{k'}$, and CB is the semiminor axis and its length is 1, and they lie respectively along the y and the x axes. The dotted curve outside is an ellipse on the same axes, and we see that the sextic curve differs very little from an ellipse.*

III.

Legendre, as we have seen, showed that, when the modulus is $\frac{1}{\sqrt{2}}$, $F(k, \phi)$ is represented by a lemniscate. SERRET (*Liouville's Journal*, vol. viii., p. 145) extended this by proving that, *whatever the modulus*, elliptic integrals of the first kind are represented by arcs of the cassinian oval, of which the lemniscate is only a particular case.

The equation of the cassinian in polar coordinates with centre as origin is

$$(1) \qquad r^4 - 2a^2r^2\cos 2\theta + a^4 - b^4 = 0,$$

$2a$ being the distance between the two foci, and b^2 the product of the distances d and d' of any point on the curve from the foci.

There are three cases accordingly as $b \gtreqless a$.

First case.

If $b = a$, then $dd' = b^2 = a^2$, and $d + d' = 2a$, and (1) is the equation of the lemniscate studied by Legendre.

Second case.

If $b < a$, then $dd' = b^2 < a^2$, and $d + d' < 2a$. Hence the curve consists of two loops equal to each other. (Fig. 12.)

---

* The case $k^2 = \frac{1}{2}$ is generally solved, as we have seen, by the lemniscate which is only of the fourth degree, and its arcs express the integral $F(k, \phi)$ *without any additional algebraical quantity*. If we take the solution of this case given by (3) we find that the equation is

$$y^2 = 2 - \tfrac{1}{2}x^2 - \tfrac{3}{2}x^{\frac{4}{3}}$$

Athough this curve is not so simple as the lemniscate it has the advantage of differing little from an ellipse.

Put $\frac{b^2}{a^2}=\sin 2\phi$, so that 2ϕ is the angle between the tangents drawn from the centre.*

If p be the length of the perpendicular from the centre on a tangent, and r the radius vector to the point of contact, then from (1) it can be shown that the p, r equation of the cassinian with centre as origin is

$$p=\frac{r^4+b^4-a^4}{2b^2r}.$$

But for the rectification of any curve we have

$$s=\int\frac{r^2d\theta}{p}.$$

Hence in this case

$$s=b^2\int\frac{2r^3d\theta}{r^4+b^4-a^4}. \tag{2}$$

Solving (1), after substituting $a^4\sin^2 2\phi$ for b^4, we get

$$r=a(\cos 2\theta\pm\sqrt{\cos^2 2\theta-\cos^2 2\phi})^{\frac{1}{2}}$$

$$\therefore\ (2)\text{ gives }\quad s=b^2\int\frac{2a^3(\cos 2\theta\pm\sqrt{\cos^2 2\theta-\cos^2 2\phi})^{\frac{3}{2}}d\theta}{a^4(\cos 2\theta\pm\sqrt{\cos^2 2\theta-\cos^2 2\phi})^2+b^4-a^4}$$

Substituting for b^4 and simplifying we get

$$s=\frac{b^2}{a}\int\frac{(\cos 2\theta\pm\sqrt{\cos^2 2\theta-\cos^2 2\phi})^{\frac{1}{2}}d\theta}{\sqrt{\cos^2 2\theta-\cos^2 2\phi}} \tag{3}$$

If we integrate between θ_0 and θ, we see that owing to the double sign in the numerator the radius vectors corresponding to these initial angles will determine upon the curve two arcs which we may represent by $s(\theta_0,\ \theta)$ and $\sigma(\theta_0,\ \theta)$, or by $s(\theta)$ and $\sigma(\theta)$ if we integrate between 0 and θ.

---

* This is perhaps most easily seen by noticing that the condition for tangency is that (1) have equal roots, that is $a^4\cos^2 2\theta=a^4-b^4$, where 2θ is the angle between the tangents. Hence

$$\begin{aligned}b^4&=a^4(1-\cos^2 2\theta)\\&=a^4\sin^2 2\theta\end{aligned}$$

$\therefore\ \frac{b^2}{a^2}=\sin 2\theta$. But by hypothesis $\frac{b^2}{a^2}=\sin 2\phi$. Therefore $2\phi=2\theta$, *i.e.*, 2ϕ is equal to the angle between the tangents drawn from the centre.

Hence from (3) we have

$$(4) \quad s(\theta_0,\ \theta) = \frac{b^2}{a}\int_{\theta_0}^{\theta} \frac{(\cos 2\theta + \sqrt{\cos^2 2\theta - \cos^2 2\phi})^{\frac{1}{2}} d\theta}{\sqrt{\cos^2 2\theta - \cos^2 2\phi}}$$

$$(5) \quad \sigma(\theta_0,\ \theta) = \frac{b^2}{a}\int_{\theta_0}^{\theta} \frac{(\cos 2\theta - \sqrt{\cos^2 2\theta - \cos^2 2\phi})^{\frac{1}{2}} d\theta}{\sqrt{\cos^2 2\theta - \cos^2 2\phi}}$$

Whence we get

$$(6) \quad s(\theta_0,\ \theta) + \sigma(\theta_0,\ \theta) = \frac{2^{\frac{1}{2}} b^2}{a}\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\cos 2\theta - \cos 2\phi}}$$

$$(7) \quad s(\theta_0,\ \theta) - \sigma(\theta_0,\ \theta) = \frac{2^{\frac{1}{2}} b^2}{a}\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\cos 2\theta + \cos 2\phi}}$$

If in (6) we put

$$(8) \qquad \sin\theta = \sin\phi \sin\chi,$$

and in (7) put

$$(9) \qquad \sin\theta = \cos\phi \sin\psi$$

then from (6) and (7) respectively we obtain

$$(10) \quad s(\theta_0,\ \theta) + \sigma(\theta_0,\ \theta) = \frac{b^2}{a}\int_{\chi_0}^{\chi} \frac{d\chi}{\sqrt{1 - \sin^2\phi \sin^2\chi}}$$

$$(11) \quad s(\theta_0,\ \theta) - \sigma(\theta_0,\ \theta) = \frac{b^2}{a}\int_{\psi_0}^{\psi} \frac{d\psi}{\sqrt{1 - \cos^2\phi \sin^2\psi}}$$

If we make $\theta_0 = 0$, we see from the relations (8) and (9) that

$$\chi_0 = 0 \text{ and } \psi_0 = 0$$

∴ (10) and (11) become respectively

$$(12) \quad \frac{a}{b^2}[s(\theta) + \sigma(\theta)] = \int_0^{\chi} \frac{d\chi}{\sqrt{1 - \sin^2\phi \sin^2\chi}} = F(\sin\phi,\ \chi)$$

$$(13) \quad \frac{a}{b^2}[s(\theta) - \sigma(\theta)] = \int_0^{\psi} \frac{d\psi}{\sqrt{1 - \cos^2\phi \sin^2\psi}} = F(\cos\phi,\ \psi)$$

But $\sin 2\phi = \frac{b^2}{a^2}$, whence it can easily be shown that

$$\sin\phi = \tfrac{1}{2}\sqrt{1+\frac{b^2}{a^2}} - \tfrac{1}{2}\sqrt{1-\frac{b^2}{a^2}}$$

$$\cos\phi = \tfrac{1}{2}\sqrt{1+\frac{b^2}{a^2}} + \tfrac{1}{2}\sqrt{1-\frac{b^2}{a^2}}$$

Hence the moduli, $\sin\phi$ and $\cos\phi$, of the elliptic integrals on the right hand of (12) and (13) are complementary. *

Third case.

If $b>a$, then $dd'=b^2>a^2$, and $d+d'>2a$, and the cassinian takes the form shown in Figure 13.

This case might be discussed in exactly the same way as the foregoing; we need however only state the results.

In this instance we put $\frac{a^2}{b^2}=\sin 2\phi$

We then get two equations corresponding to (6) and (7), viz.,

$$s(\theta_0, \theta)+\sigma(\theta_0, \theta)=\frac{2^{\frac{1}{2}}b^2}{a}\int_{\theta_0}^{\theta}\frac{(\cot\text{an}\,2\phi+\sqrt{\cos^2 2\theta+\cot\text{an}^2 2\phi})^{\frac{1}{2}}}{\sqrt{\cos^2 2\theta+\cot\text{an}^2 2\phi}}\,d\theta$$

$$s(\theta_0, \theta)-\sigma(\theta_0, \theta)=\frac{2^{\frac{1}{2}}b^2}{a}\int_{\theta_0}^{\theta}\frac{(-\cot\text{an}\,2\phi+\sqrt{\cos^2 2\theta+\cot\text{an}^2 2\phi})^{\frac{1}{2}}}{\sqrt{\cos^2 2\theta+\cot\text{an}^2 2\phi}}\,d\theta$$

In this case the relations connecting θ, χ, and ψ corresponding to (8) and (9) are

(13a) $$\sqrt{\cos^2 2\theta+\cot\text{an}^2 2\phi}=\frac{1-2\sin^2\phi\sin^2\chi}{\sin 2\phi}$$

$$\sqrt{\cos^2 2\theta+\cot\text{an}^2 2\phi}=\frac{1-2\cos^2\phi\sin^2\psi}{\sin 2\phi}$$

* Of course all this applies equally to the case $b=a$, for then $\phi=\frac{\pi}{4}$, in which case the two loops meet at the centre and we get the lemniscate, and from (5) we see that the arc represented by $\sigma(\theta_0, \theta)$ disappears as it ought to do since the polar equation of the lemniscate is only of the second degree in r.

Then the results corresponding to (12) and (13) are

$$(14)\qquad \frac{1}{b}[s(\theta)+\sigma(\theta)]=\mathrm{F}(\sin\phi,\ \chi)$$

$$(15)\qquad \frac{1}{b}[s(\theta)-\sigma(\theta)]=\mathrm{F}(\cos\phi,\ \psi)$$

where the moduli $\sin\phi$, $\cos\phi$ are again complementary, viz.,

$$\sin\phi=\tfrac{1}{2}\sqrt{1+\frac{a^2}{b^2}}-\tfrac{1}{2}\sqrt{1-\frac{a^2}{b^2}}$$

$$\cos\phi=\tfrac{1}{2}\sqrt{1+\frac{a^2}{b^2}}+\tfrac{1}{2}\sqrt{1-\frac{a^2}{b^2}}$$

From (12), (13), (14), and (15) we see that in every case and *whatever the modulus, the elliptic integral of the first kind can be represented by the sum or difference of two arcs of a cassinian oval,* and conversely (by addition and substraction of (12) and (13), or of (14) and (15)) *the arc of the cassinian oval is represented by the sum or difference of two elliptic integrals of the first kind whose moduli are complementary.**

IV.

The next to attack the problem we are investigating was W. Roberts of Dublin (*Liouville's Journal*, vol. viii., p. 263, vol. ix., p. 155, vol. x., p. 297).

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* Comparing Serret's result with Legendre's in the last section, it should be observed that if in (8) we make $\theta=\phi$, then $\chi=\frac{\pi}{2}$, hence, the amplitude being a right angle, we may write $s(\theta)+\sigma(\theta)=s_1$ where s_1 is as before one-fourth of the total length of the curve. Then from (12) we get

$$\frac{a}{b^2}s_1=\mathrm{F}\left(\sin\phi,\ \frac{\pi}{2}\right)$$

Similarly, if in (13a) we put $\theta=\frac{\pi}{4}$ $\left(\text{and of course } \sin 2\phi=\frac{a^2}{b^2}\right)$ then $\chi=\frac{\pi}{2}$, and we then as above get from (14)

$$\frac{s_1}{b}=\mathrm{F}\left(\sin\phi,\ \frac{\pi}{2}\right)$$

From these two equations we see that, just as in Legendre's solution, by properly choosing the ends of the arc, *i.e.*, by integrating between suitable limits, the elliptic integral of the first kind *for any modulus and without any extra algebraical quantity* is represented by the arc of a cassinian oval.

He showed that the curves in which a sphere is cut by a cone of the second order, one of whose principal external axes is a diameter of the sphere, can be rectified by means of an expression containing all three kinds of elliptic integrals. Taking the particular case in which the elliptic integrals of the second and third kinds vanished he got a single curve on the surface of the sphere, whose arc reckoned from the extremity of the semi-axis was expressed by an elliptic integral of the first kind. He showed that the form of this curve was similar to that of the lemniscate, and that it was the locus of points on the sphere the product of whose distances from two fixed points on the sphere was constant.* Hence Kiepert called this curve, discovered by Roberts in 1843, the "spherical lemniscate."

V.

SERRET returned to the problem of the geometrical representation of the first elliptic integral, and wrote an important Memoir on the subject (*Liouville's Journal*, vol. x., p. 257) and an Additional Note (p. 286). A new method of solution was suggested to him by the fact that the equation of the lemniscate

$$(x^2+y^2)^2 - a^2(x^2-y^2) = 0$$

is satisfied by

$$x = a\frac{z+z^3}{1+z^4}, \qquad y = a\frac{z-z^3}{1+z^4}$$

whence (1) $$ds = \sqrt{dx^2+dy^2} = \frac{a\sqrt{2}\,dz}{\sqrt{1+z^4}}$$

so that the arc is expressible as an elliptic integral.†

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* If the radius of the sphere were made infinitely great it would become a plane, and the curve on it would become a plane lemniscate.

† This leads to exactly the same solution as Legendre's on p. 7, for let

$$\sin\phi = \frac{1-z^2}{\sqrt{1+z^4}}, \text{ and therefore } \cos\phi = \frac{z\sqrt{2}}{\sqrt{1+z^4}}$$

$$\text{and } \sqrt{1-\tfrac{1}{2}\sin^2\phi} = \frac{1}{\sqrt{2}}\,\frac{1+z^2}{\sqrt{1+z^4}}$$

Whence, after differentiating and reducing, we get

$$\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}} = \frac{2dz}{\sqrt{1+z^4}}$$

But $$s = a\sqrt{2}\int\frac{dz}{\sqrt{1+z^4}} = \frac{a}{\sqrt{2}}\int\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}} = \frac{a}{\sqrt{2}}\,\mathrm{F}\left(\frac{1}{\sqrt{2}}, \phi\right)$$

By generalising this method he was able to prove that not merely the lemniscate and the cassinian but *an infinite number* of plane curves represented the first elliptic transcendental.

In the generalisation of equation (1) he discusses

$$(1a) \qquad ds^2 = dx^2 + dy^2 = C^2 \frac{dz^2}{P}$$

where P is a quartic rational function of z, and C a constant.

Of course P does not include multiple factors else, after extracting the square root of each side, (1a) would not lead to an elliptic integral.

Nor can P have real factors, for since

$$(2) \qquad dx + i\,dy \quad . \quad dx - i\,dy = C^2 \frac{dz^2}{P}$$

we see that every real value of z which would make the right-hand side of (2) infinite would necessarily make both $dx + i\,dy$ and $dx - i\,dy$ infinite, and this is impossible since P has not multiple factors.

Therefore P can only have four imaginary factors, and, since their product is real, they must be conjugate two and two, say b and β conjugate and c and γ conjugate. Then (1a) becomes

$$(3) \qquad ds^2 = dx^2 + dy^2 = \frac{C^2 dz^2}{(z-b)(z-\beta)(z-c)(z-\gamma)}$$

We may therefore let $P = p . \pi$, where p and π are two conjugate functions each of the second degree in z. Therefore (2) gives

$$(4) \qquad \frac{dx + i\,dy}{C\frac{dz}{p}} \quad . \quad \frac{dx - i\,dy}{C\frac{dz}{\pi}} = 1$$

The two functions on the left hand side are, therefore, conjugate and have 1 for modulus. Serret goes on to prove in section II. of his Memoir that these conjugate quantities must have the form

$$\frac{dx + i\,dy}{C\frac{dz}{p}} = \frac{\Delta r^2}{D\rho^2} \quad \text{and} \quad \frac{dx - i\,dy}{C\frac{dz}{\pi}} = \frac{D\rho^2}{\Delta r^2}$$

or (5) $$x + iy = C\int \frac{\Delta r^2\,dz}{p . D\rho^2} \quad \text{and} \quad x - iy = C\int \frac{D\rho^2\,dz}{\pi . \Delta r^2}$$

where r is any integral function of the variable z, D the G.C.M. of r and its first derivative, and ρ and Δ the conjugate complex quantities of r and D respectively.

But by a real and rational substitution

$$\frac{dz^2}{(z-b)(z-\beta)(z-c)(z-\gamma)} \text{ may be transformed into } \frac{dz_1^{\,2}}{(z_1^{\,2}-a^2)(z_1^{\,2}-\alpha^2)}$$

where a and α are conjugate complex quantities.

Hence we may write (3) as follows

$$(6) \qquad ds^2 = dx^2 + dy^2 = \frac{C^2 dz^2}{(z^2-a^2)(z^2-\alpha^2)}$$

and $\therefore$ $\qquad p = z^2 - a^2, \qquad \pi = z^2 - \alpha^2.$

For the success of Serret's method it was necessary to choose r and ρ so that they contained no factors except those of p and π respectively. Wherefore he put

$$(7) \quad \begin{cases} r = (z-a)^m(z+a)^n, & D = (z-a)^{m-1}(z+a)^{n-1} \\ \rho = (z-\alpha)^m(z+\alpha)^n, & \Delta = (z-\alpha)^{m-1}(z+\alpha)^{n-1}. \end{cases}$$

Therefore from (5)

$$(8) \qquad x + iy = C\int \frac{(z-a)^m(z+a)^n\, dz}{(z-a)^{m+1}(z+a)^{n+1}} = C\int f(z)\, dz, \text{ say}$$

For shortness let

$$(9) \qquad \begin{cases} \phi(z) = f(z)(z-a)^{m+1} = \dfrac{(z-a)^m(z+a)^n}{(z+a)^{n+1}} \\[2ex] \psi(z) = f(z)(z+a)^{n+1} = \dfrac{(z-a)^m(z+a)^n}{(z-a)^{m+1}} \end{cases}$$

Now $\qquad f(z) = \dfrac{(z-a)^m(z+a)^n}{(z-a)^{m+1}(z+a)^{n+1}}$

Therefore by partial fractions

$$(10) \quad f(z) = \frac{\phi(a)}{(z-a)^{m+1}} + \frac{\phi'(a)}{1!(z-a)^m} + \quad \cdots\cdots \quad + \frac{\phi^{(m)}(a)}{m!(z-a)}$$
$$+ \frac{\psi(-a)}{(z+a)^{n+1}} + \frac{\psi'(-a)}{1!(z+a)^n} + \quad \cdots\cdots \quad + \frac{\psi^{(n)}(-a)}{n!(z+a)}$$

We wish to find the condition that the curve determined by x and y in (8) be *algebraic.*

Since $x+iy=C\int f(z)dz$ we see from (10) that in order that x and y be purely algebraic and not contain a logarithmic part the conditions are

(11) $$\phi^{(m)}(a)=0 \text{ and } \psi^{(n)}(-a)=0.$$

But from (10) by multiplying up we get

$f(z)(z-a)^{m+1}(z-a)^{n+1}$ or $(z-a)^m(z+a)^n$

$$=(z+a)^{n+1}\left\{\phi(a)+\frac{\phi'(a)}{1!}(z-a)+\quad . \quad . \quad . \quad +\frac{\phi^{(m)}(a)}{m!}(z-a)^m\right\}$$

$$+(z-a)^{m+1}\left\{\psi(-a)+\frac{\psi'(-a)}{1!}(z+a)+\quad . \quad . \quad . \quad +\frac{\psi^{(n)}(-a)}{n!}(z+a)^n\right\}$$

Arranging the right-hand side according to powers of z we get

$$(z-a)^m(z+a)^n=\left\{\frac{\phi^{(m)}(a)}{m!}+\frac{\psi^{(n)}(-a)}{n!}\right\}z^{m+n+1}+\quad . \quad . \quad . \quad .$$

wherefore we see that

(12) $$\frac{\phi^{(m)}(a)}{m!}+\frac{\psi^{(n)}(-a)}{n!}=0$$

which shows that if one of the conditions in (11) be satisfied the other will necessarily follow.

Hence the sufficient and necessary condition that the curves determined by x and y in

$$x+iy=C\int\frac{(z-a)^m(z+a)^n dz}{(z-a)^{m+1}(z+a)^{n+1}}$$

be algebraic is $\phi^{(m)}(a)=0$. And the arcs of those curves are elliptic integrals of the first kind, for

$$dx+i\,dy=C\frac{(z-a)^m(z+a)^n dz}{(z-a)^{m+1}(z+a)^{n+1}}$$

$\therefore$ also $$dx-i\,dy=C\frac{(z-a)^m(z+a)^n dz}{(z-a)^{m+1}(z+a)^{n+1}}$$

Wherefore, by multiplication,

$$ds^2 = dx^2 + dy^2 = \frac{C^2 dz^2}{(z^2 - a^2)(z^2 - \alpha^2)}$$

or

$$s = C\int \frac{dz}{\sqrt{(z^2 - a^2)(z^2 - \alpha^2)}},$$

which is an elliptic integral of the first kind.

Also, since m and n may be anything whatever, we have a *double infinity of curves*, analogous to the lemniscate, represented by (8), and since the arcs of those curves are elliptic integrals of the first kind they can be added, subtracted, multiplied, and divided just as the arcs of the circle and lemniscate can.

After performing the integration in (8) and substituting the values of a and α that satisfy the conditions, we equate the real and imaginary parts on each side and so obtain the coordinates x and y of the required curve as functions of the parameter z.

We shall take two examples to illustrate this infinite group of curves discovered by Serret.

First Example.

If $m = 1$, n being any integer, then the condition that the curve be algebraic is $\phi'(\alpha) = 0$.

Now, from (9) we see that

$$\phi(z) = \frac{(z - a)(z + \alpha)^n}{(z + a)^{n+1}}$$

whence, obtaining $\phi'(z)$ and putting α instead of z and equating to zero, we get

$$(13) \qquad \frac{a^2 + \alpha^2}{a\alpha} = 2\frac{n - 1}{n + 1}$$

But, if we reduce

$$C\int \frac{dz}{\sqrt{(z^2 - a^2)(z^2 - \alpha^2)}} \text{ to the standard form } C\int \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}},$$

we find that k^2, the square of the modulus, is $= \dfrac{(a + \alpha)^2}{4a\alpha}$

wherefore, from (13)

$$k^2 = \frac{(a+\alpha)^2}{4a\alpha} = \frac{n}{n+1}$$

Hence if we take a and its conjugate α so that they satisfy (13), then the curves defined by

$$(14) \qquad x+iy = \mathrm{C}\int \frac{(z-a)(z+\alpha)^n dz}{(z-a)^2(z+\alpha)^{n+1}}$$

will be algebraic, and their arcs will be represented by the integral $\mathrm{C}\int \frac{dz}{\sqrt{(z^2-a^2)(z^2-\alpha^2)}}$ whose modulus k is $= \sqrt{\frac{n}{n+1}}$; and, as n may be any integer whatever, we have an *infinite number* of such curves.*

SECOND EXAMPLE.

Next take $m=2$. Then the condition that the curves be algebraic is $\phi''(a)=0$.

But $\phi(z) = \frac{(z-a)^2(z+a)^n}{(z+\alpha)^{n+1}}$. Differentiating twice and putting a for z, $\phi''(a)=0$ gives

$$(15) \qquad \frac{a^2+\alpha^2}{a\alpha} = \frac{2(n+1)(n-2) \pm 4\sqrt{2n(n+1)}}{(n+1)(n+2)}.$$

If a and α satisfy this condition, then the arcs of the algebraic curves defined by

$$(16) \qquad x+iy = \mathrm{C}\int \frac{(z-a)^2(z+\alpha)^n dz}{(z-a)^3(z+\alpha)^{n+1}}$$

will be represented by the elliptic integral $\mathrm{C}\int \frac{dz}{\sqrt{(z^2-a^2)(z^2-\alpha^2)}}$,

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* If we make $n=1$ then $k=\frac{1}{\sqrt{2}}$, *i.e., the lemniscate is the simplest case of this infinite class of curves.* To obtain its equation we may proceed thus:— If $n=1$ then from (13) $a^2+\alpha^2=0$ and $a\alpha=1$, whence $a^2=i$ and $\alpha^2=-i$. Putting these values in (14) we get, after one or two steps,

$$x+iy = \mathrm{C}\frac{z^3+iz}{z^4+1}$$

whence $x=\mathrm{C}\frac{z^3}{z^4+1}$, $y=\mathrm{C}\frac{z}{z^4+1}$. Eliminating z between these, we get $(x^2+y^2)^2 = \mathrm{C}^2xy$.

the square of whose modulus is from (15)

$$k^2=\frac{(a+\alpha)^2}{4a\alpha}=\frac{n(n+1)\pm\sqrt{2n(n+1)}}{(n+1)(n+2)}.$$

If we also make $n=2$, then (15) gives

$$a^2+\alpha^2=\frac{2}{\sqrt{3}},\text{ and } a\alpha=1$$

whence
$$a^2=\frac{1+i\sqrt{2}}{\sqrt{3}},\text{ and } \alpha^2=\frac{1-i\sqrt{2}}{\sqrt{3}}.$$

Putting these values in (16) we get ultimately

$$x+iy=\mathrm{C}\frac{z^3-\frac{5-i\sqrt{2}}{3\sqrt{3}}z}{\left(z^2-\frac{1-i\sqrt{2}}{\sqrt{3}}\right)^2}.$$

On rationalising the denominator and then equating the real and imaginary parts on each side we obtain for the coordinates

$$x=\mathrm{C}\frac{z^7-\frac{11z^5}{3\sqrt{3}}+\frac{11z^3}{9}+\frac{z}{9\sqrt{3}}}{\left(z^4-\frac{2}{\sqrt{3}}z^2+1\right)^2}.$$

$$y=\mathrm{C}\frac{\frac{5\sqrt{2}}{3\sqrt{3}}z^5-\frac{14\sqrt{2}}{9}z^3+\frac{11\sqrt{2}z}{9\sqrt{3}}}{\left(z^4-\frac{2}{\sqrt{3}}z^2+1\right)^2}$$

The curve is of the sixth degree, and, in an Additional Note to his Memoir, Serret shows that its equation in polar coordinates is

$$9r^2(r^2-2\mathrm{C}^2\cos2\theta)^2+8\mathrm{C}^4(r^2-2\mathrm{C}^2\cos2\theta)+\frac{16\mathrm{C}^6}{9\sqrt{3}}=0$$

where 2C (*cf.* equations (1) and (1^a)) is the distance between the two foci of the curve.*

VI.

As we have said on p. 16, Serret could only satisfy the conditions that the curve be algebraic if r contained no other factors than those of p, viz., $z-a$ and $z+a$. He said, "En général pour une forme déterminée des polynômes r et ρ les conditions ne pouront subsister en même temps; mais il existe un cas très étendu, où elles pourront toujours être satisfaites, c'est celui, où les polynômes r et ρ ne renfermeront que les facteurs linéaires des polynômes données p et π."

L. Kiepert in a dissertation entitled *De curvis quarum arcus integralibus ellipticis primi generis exprimuntur* (Berlin, 1870) showed that this restriction was not necessary, and that therefore

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* We have always taken the condition $\phi^{(m)}(\alpha)=0$, but of course we take the one or the other of the conditions (11) according as m or n is the smaller. If the smaller exceeds 2 then the condition for a and α will be at least of the third degree and cannot in general be resolved, but this does not affect the reasoning by which we obtained an infinity of curves for any given value of the smaller of m or n.

Moreover, n has throughout been supposed *integral*, but M. Liouville proved in *Liouville's Journal*, vol. x. p. 293, that *n need not be integral but only rational* in order that Serret's infinity of curves for x and y remain algebraic.

We have seen that $\phi^{(m)}(\alpha)=0$ leads always to a relation symmetrical and homogeneous in a and α. Serret shows that this relation in its most general form is

$$\sum_{p=0}^{m}\sum_{q=0}^{n}(-1)^{p+q}\frac{(n+m-p+1)!\,(n+m-q+1)!}{(m-p+1)!(m-q+1)!(p+1)!(q+1)!(n+m-p-q+1)!} \times (2\alpha)^p(2a)^q(\alpha+a)^{2m-p-q}=0$$

Cayley (*Elliptic Functions*, chap. xv.) states the relation between a and α much more briefly thus:—Putting ξ for $\dfrac{(a+\alpha)^2}{4a\alpha}$, the square of the modulus, then the relation is

$$\frac{1}{\xi^{n-m}}\left(\frac{d}{d\xi}\right)^m \xi^n(\xi-1)^m=0$$

If we make $m=1$ or 2, etc., then this gives the same values of the modulus as we have obtained above.

Serret's class of curves could be greatly extended. For let $b_1, b_2, \ldots b_x$ be other x imaginaries, and $\beta_1, \beta_2, \ldots \beta_x$ their conjugates. Also let

$$r = (z-a)^m (z+a)^n (z-b_1)^{n_1} (z-b_2)^{n_2} \quad \ldots \quad (z-b_x)^{n_x}$$

Then as in V. (7) we will get corresponding equations for ρ, D and Δ. Also as before let $p = z^2 - a^2$, $\pi = z^2 - a^2$.

Then corresponding to V. (8) we get

$$(1)\ x + iy = \int \frac{\Delta r^2\, dz}{p.\,\mathrm{D}\rho^2}$$

$$= \int \frac{(z-a)^m (z+a)^n (z-b_1)^{n_1+1} (z-b_2)^{n_2+1} \ldots\ldots (z-b_x)^{n_x+1} dz}{(z-a)^{m+1} (z+a)^{n+1} (z-\beta_1)^{n_1+1} (z-\beta_2)^{n_2+1} \ldots (z-\beta_x)^{n_x+1}}$$

$$= \int f(z) dz, \text{ say.}$$

Also V. (9) now becomes

$$(2) \quad \begin{cases} \phi(z) = f(z)(z-a)^{m+1} \\ \psi(z) = f(z)(z+a)^{n+1} \\ \chi_1(z) = f(z)(z-\beta_1)^{n_1+1} \\ \ldots\ldots\ldots \\ \chi_x(z) = f(z)(z-\beta_x)^{n_x+1} \end{cases}$$

Instead of V. (10) we have

$$\begin{aligned} f(z) = {} & \frac{\phi(a)}{(z-a)^{m+1}} + \frac{\phi'(a)}{1!(z-a)^m} + \ldots + \frac{\phi^{(m)}(a)}{m!(z-a)} \\ & + \frac{\psi(-a)}{(z+a)^{n+1}} + \frac{\psi'(-a)}{1!(z+a)^n} + \ldots + \frac{\psi^{(n)}(-a)}{n!(z+a)} \\ & + \frac{\chi_1(\beta_1)}{(z-\beta_1)^{n_1+1}} + \frac{\chi_1'(\beta_1)}{1!(z-\beta_1)^{n_1}} + \ldots + \frac{\chi_1^{(n_1)}(\beta_1)}{n_1!(z-\beta_1)} \\ & + \ldots\ldots\ldots\ldots \\ & + \frac{\chi_x(\beta_x)}{(z-\beta_x)^{n_x+1}} + \frac{\chi'_x(\beta_x)}{1!(z-\beta_x)^{n_x}} + \ldots + \frac{\chi_x^{(n_x)}(\beta_x)}{n_x!(z-\beta_x)}. \end{aligned}$$

On multiplying up and comparing the powers of z on the left and right-hand side of this we get as in V. (12)

$$(3)\quad \frac{\phi^{(m)}(a)}{m!}+\frac{\psi^{(n)}(-a)}{n!}+\frac{\chi_1{}^{(n_1)}(\beta_1)}{n_1!}+\ldots\ldots+\frac{\chi_x{}^{(n_x)}(\beta_x)}{n_x!}=0$$

In order that x and y be purely algebraic we get $x+2$ conditions corresponding to V. (11), viz.,

$$\phi^{(m)}(a)=0,\quad \psi^{(n)}(-a)=0,\quad \chi_1{}^{(n_1)}(\beta_1)=0\ \ldots\ldots\ \chi_x{}^{(n_x)}(\beta_x)=0.$$

But because of (3) these reduce to $x+1$ conditions, which can always be fulfilled for we have $x+1$ quantities at our disposal, viz.,

$a,\ b_1,\ b_2,\ \ldots\ldots\ b_x$

or their conjugates $\alpha,\ \beta_1,\ \beta_2,\ \ldots\ldots\ \beta_x$.

Thus the conditions that the curves represented by (1) be algebraic can be fulfilled, and since p and π are, as we have said, the same as before, the arcs of these curves will be represented as formerly by an elliptic integral of the first kind, viz.,

$$s=\int\frac{dz}{\sqrt{(z^2-a^2)(z^2-\alpha^2)}}.$$

Example.

Kiepert does not give a direct example of (1) we shall therefore fully discuss the following case.

Let $\qquad r=(z-a)(z+a)(z-b)^n$

.'. from (1)

$$(4)\qquad x+iy=\int\frac{(z-a)(z+a)(z-b)^{n+1}}{(z-\alpha)^2(z+\alpha)^2(z-\beta)^{n+1}}dz$$

and from (2) we get

$$\phi(z)=\frac{(z-a)(z+a)(z-b)^{n+1}}{(z+\alpha)^2(z-\beta)^{n+1}}$$

$$\psi(z)=\frac{(z-a)(z+a)(z-b)^{n+1}}{(z-\alpha)^2(z-\beta)^{n+1}}$$

$$\chi(z)=\frac{(z-a)(z+a)(z-b)^{n+1}}{(z-\alpha)^2(z+\alpha)^2}.$$

That the curves (4) be algebraic the conditions are

$$\phi'(a)=0, \qquad \psi'(-\alpha)=0.$$

Now $$\frac{\phi'(z)}{\phi(z)}=\frac{1}{z-a}+\frac{1}{z+a}-\frac{2}{z+\alpha}+(n+1)\left\{\frac{1}{z-b}-\frac{1}{z-\beta}\right\}$$

$$=\frac{2za+2a^2}{(z^2-a^2)(z+\alpha)}+\frac{(n+1)(b-\beta)}{(z-b)(z-\beta)}.$$

Hence the condition $\phi'(a)=0$ evidently gives

(5) $(a^2+a^2)\{a^2-a(b+\beta)+b\beta\}+(n+1)a(a^2-a^2)(b-\beta)=0.$

The second condition, $\psi'(-\alpha)=0$, will obviously be got from this by putting $-\alpha$ instead of a, therefore

(6) $(a^2+a^2)\{a^2+a(b+\beta)+b\beta\}-(n+1)a(a^2-a^2)(b-\beta)=0.$

Adding (5) and (6) we get

$$(a^2+a^2)(a^2+b\beta)=0$$

the conjugate of which is $(a^2+a^2)(a^2+b\beta)=0.$

By subtraction we get

$$(a^2+a^2)(a^2-a^2)=0.$$

In order that this equation be fulfilled we must have either

$$a^2+a^2=0 \text{ or } a^2-a^2-0.$$

We shall examine each of those conditions for the curve being algebraic.

First case, $a^2+a^2=0$.

(5) then gives $\qquad (a^2-a^2)(b-\beta)=0$

hence either $\qquad a^2-a^2=0$ or $b-\beta=0$.

The first of these is impossible for taken along with $a^2+a^2=0$, it would give zero for a and a.

We therefore must take $b-\beta=0$, or $b=\beta$, so that the curve (4) is

$$x+iy=\int\frac{(z-a)(z+a)}{(z-a)^2(z+a)^2}dz$$

which, see the foot-note on page 19, is the lemniscate.

Second case, $a^2 - a^2 = 0$ or $a^2 = a^2$.

But the arc $$s = \int \frac{dz}{\sqrt{(z^2 - a^2)(z^2 - a^2)}} = \int \frac{dz}{z^2 - a^2}$$

which is no longer an elliptic integral of the first kind, and therefore we need not further discuss the case.

VII.

The most general solutions, however, of the problem we are investigating have been obtained by employing the Weierstrassian notation for elliptic functions.

The transition from the older notation to that now almost universally employed can be very briefly stated for our present purpose.

Call the arc of a curve u, then up to the present we have used the relation

$$\sqrt{dx^2 + dy^2} = du = \frac{dz}{\sqrt{a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4}}$$

But the elliptic function z of the argument u defined as above by the equation

$$\left(\frac{dz}{du}\right)^2 = a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4$$

can be transformed into Weierstrass's elliptic function $\mathrm{p}u$ of the argument u defined by the equation

$$(1) \qquad (\mathrm{p}'u)^2 \text{ or } \left(\frac{d.\mathrm{p}u}{du}\right)^2 = 4\mathrm{p}^3 u - g_2 \mathrm{p}u - g_3$$

($\mathrm{p}^3 u$ denotes the cube of $\mathrm{p}u$) where g_2 and g_3 are the *invariants* of the 2nd and 3rd degrees of the quadric function

$$a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4.$$

If $$\int \sqrt{dx^2 + dy^2} = u = \int \frac{d(\mathrm{p}u)}{\sqrt{4\mathrm{p}^3 u - g_2 \mathrm{p}u - g_3}}$$

then the arc u of the curve is, in the new notation, an elliptic integral of the first kind.

Furthermore, if 2ω and $2\omega'$ be the two fundamental real and imaginary periods respectively of this doubly periodic function $\wp u$, and $2\omega''$ their sum, and if we put e_1 for $\wp\omega$, e_2 for $\wp\omega''$, and e_3 for $\wp\omega'$, then

$$(\wp' u)^2 = 4(\wp u - e_1)(\wp u - e_2)(\wp u - e_3).$$

Comparing this with (1) we see that

$$\begin{aligned} 0 &= e_1 + e_2 + e_3 \\ g_2 &= -4(e_2 e_3 + e_3 e_1 + e_1 e_2) \\ g_3 &= 4 e_1 e_2 e_3. \end{aligned}$$

From $\wp u$ we define the σu function of Weierstrass by the equation

$$\frac{d^2 \log \sigma u}{du^2} = -\wp u$$

with the additional equations

$$\sigma(0) = 0,\ \sigma'(0) = 1,\ \sigma''(0) = 0$$

for the determination of the constants of integration.*

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* The best summary of Weierstrass's functions is contained in H. A. Schwarz's *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen, nach Vorlesungen und Aufzeichnungen des Herrn K. Weierstrass* (2nd edition, 1893). The theory of these functions is developed in Halphen's *Traité des Fonctions Elliptiques.* Weierstrass's Memoirs have been collected and published in *Mathematische Werke von Karl Weierstrass* (Berlin, Bd. I. 1894; Bd. II. 1895). In Bd. II., pp. 245-309, are two articles *Zur Theorie der elliptischen Functionen.* No better account of the methods of Legendre, Jacobi and Weierstrass can be found than that in Müller's edition of Enneper's *Elliptische Functionen, Theorie und Geschichte* (Halle, 1890). It contains also for the student of the subject a great mass of bibliographical details. The best Memoirs in English on Weierstrass's methods are three by A. L. Daniels in vols. vi. and vii. (1884 and 1885) of *The American Journal of Mathematics.* Dr A. R. Forsyth has a Memoir on the same subject in vol. xxii. (1887) of *The Quarterly Journal of Mathematics.* In Greenhill's *Elliptic Functions* some parts of the modern notation are developed alongside of the old. In chap. vii. of A. C. Dixon's excellent little book on *Elliptic Functions,* there is a very brief sketch of the $\wp-$ and $\zeta-$ functions (Halphen uses the symbol ζu for $\frac{d}{du}\log\sigma u$ or $\frac{\sigma'}{\sigma}u$). Harkness and Morley in chap. vii. of their *Treatise on the Theory of Functions* devote about 60 pages to the $\wp-$, $\sigma-$, and $\zeta-$ functions.

VIII.

KIEPERT* by using the Weierstrassian notation was able to obtain a more general solution than Serret's. Indeed, we shall show that Kiepert discovered a class of curves, whose arcs are elliptic integrals of the first kind, *which includes Serret's infinity of curves as a particular case.*

If $\phi\ (u)$ be a doubly periodic function of the argument u of the r^{th} degree having 2ω and $2\omega'$ as its primary periods then we shall in what follows use two methods of expressing this function by means of the $\sigma -$ function.

First,† it is always possible to find $2r+1$ quantities $C, a_1, a_2, \ldots\ldots a_r, b_1, b_2, \ldots\ldots b_r$, so that

$$(1)\quad \phi(u) = C\frac{\sigma(u-b_1)\sigma(u-b_2)\ \ldots\ldots\ \sigma(u-b_r)}{\sigma(u-a_1)\sigma(u-a_2)\ \ldots\ldots\ \sigma(u-a_r)}$$

where C is a constant, and $b_1, b_2, \ldots\ldots b_r$ are the values of u for which $\phi(u)$ vanishes, and $a_1, a_2, \ldots\ldots a_r$ are the values of u for which $\phi(u)$ is infinitely great.

It can be proved that as a consequence of (1)

$$(2)\ a_1 + a_2 + \ \ldots\ldots\ + a_r = b_1 + b_2 + \ \ldots\ldots\ + b_r.$$

Conversely, if $a_1, \ldots\ldots a_r, b_1 \ldots\ldots b_r$ satisfy equation (2), then every doubly periodic function of the r^{th} degree can be expressed as in (1).

Secondly, Kiepert makes very frequent use of another mode of expressing any doubly periodic function $\phi(u)$ of the r^{th} degree, viz., a modified form of the expression given at p. 20 of the *Formeln und Lehrsätze*

Among the values $a_1, a_2, \ldots\ldots a_r$ of u for which $\phi(u)$ is infinite there may be only m different from each other, viz.,

$$a_1,\ a_2\ \ldots\ldots\ a_m,$$

and these may occur respectively $r_1,\ r_2\ \ldots\ldots\ r_m$ times

* Inaugural Dissertation *De curvis quarum arcus integralibus ellipticis primi generis exprimuntur* (Berlin 1870); *Ueber Curven deren Bogen ein elliptisches Integral erster Gattung ist* (*Crelle's Journal*, vol. lxxix., 1875; and *Berichte der naturforschenden Gesellschaft zu Freiburg*, 1876).

† *Cf. Formeln*, p. 15, or Halphen's F. E., vol. I, p. 213.

i.e., the infinities may respectively be of the orders

$$r_1, r_2 \ldots\ldots r_m,$$

so that $$r_1+r_2+\ldots\ldots+r_m=r.$$

Now, in the development of $\phi(u)$ according to powers of $u-a$, the sum of the terms with negative exponents may be, say,

$$(3)\quad c_{1,1}(u-a_1)^{-1}+c_{1,2}(u-a_1)^{-2}+\ldots$$
$$\ldots+c_{1,r_1-1}(u-a_1)^{-r_1+1}+c_{1,r_1}(u-a_1)^{-r_1}$$

with similar series for $u-a_2$, etc., and lastly

$$c_{m,1}(u-a_m)^{-1}+c_{m,2}(u-a_m)^{-2}+\ldots$$
$$\ldots+c_{m,r_m-1}(u-a_m)^{-r_m+1}+c_{m,r_m}(u-a_m)^{-r_m}.$$

From these expressions it is evident that if we put

$$\phi(u, a_1) \quad \text{for} \quad \sum_{\nu=1}^{r_1}\frac{(-1)^{\nu-1}}{(\nu-1)!}c_{1,\nu}\frac{d^\nu\log\sigma(u-a_1)}{du^\nu}$$

$\phi(u, a_2)$, $\phi(u, a_3) \ldots\ldots \phi(u, a_m)$ having similar meanings, then

$$(4)\quad \phi(u)-\phi(u, a_1)-\phi(u, a_2)\ldots\ldots-\phi(u, a_m)=\chi(u), \text{ say,}$$

where $\chi(u)$ must by hypothesis be a function which cannot become infinite for any finite value of u.

If we write down the expanded form of (4) we see that

$$c_{1,1}+c_{2,1}+\ldots\ldots+c_{m,1}=0.$$

Also, on differentiating each side of (4) we obtain that the derivative of $\chi(u)$ is doubly periodic since the derivatives ϕ' are so. But every doubly periodic function must have infinite values, therefore $\chi'(u)$ cannot be a function of u, it can at most be a constant. If we put in turn $u+2\omega$ and $u+2\omega'$ for u in the differential of each side of (4), we shall find that this constant is zero. Hence, since $\chi'(u)$ is zero, $\chi(u)$ is constant, say, c_0.

Wherefore we may write (4) as follows:

$$\phi(u)=c_0+\phi(u, a_1)+\phi(u, a_2)+\ldots\ldots+\phi(u, a_m), \quad \text{or}$$

$$(5)\quad \phi(u)=c_0+\sum_{\nu=1}^{r_1}c_{1,\nu}\frac{d^\nu\log\sigma(u-a_1)}{du^\nu}+\sum_{\nu=1}^{r_2}c_{2,\nu}\frac{d^\nu\log\sigma(u-a_2)}{du^\nu}+\ldots$$
$$\ldots+\sum_{\nu=1}^{r_m}c_{m,\nu}\frac{d^\nu\log\sigma(u-a_m)}{du^\nu}$$

where, as we have seen,

(6) $$c_{1,1}+c_{2,1}+\ \dots\ +c_{m,1}=0.$$

Comparing the right-hand side of (5) with the expression given above for $\phi(u, a_1)$, it will be seen that we have omitted the part $\dfrac{(-1)^{\nu-1}}{(\nu-1)!}$ in the 2nd, 3rd, 4th, etc., terms. Of course this is permissible since the factor omitted only affects the coefficients.

Now, using (5), let the equation of the curves we are going to investigate be

(7) $$x+iy=c_0+\sum_{\nu=1}^{r_1} c_{1,\nu}\frac{d^\nu \log\sigma(u-a_1)}{du^\nu}+\ \dots$$

$$\dots\ +\sum_{\nu=1}^{r_m} c_{m,\nu}\frac{d^\nu \log\sigma(u-a_m)}{du^\nu}$$

with the condition (6) amongst the coefficients.

We have to find the condition that those curves be algebraic, and to find a method of obtaining the modulus of the elliptic integral of the first kind which the arcs of the curves represent.

Differentiating (7) we get

(8) $$\frac{dx+idy}{du}$$

$$=f(u)=\sum_{\nu=1}^{r_1} c_{1,\nu}\frac{d^{\nu+1}\log\sigma(u-a_1)}{du^{\nu+1}}+\ \dots\ +\sum_{\nu=1}^{r_m} c_{m,\nu}\frac{d^{\nu+1}\log\sigma(u-a_m)}{du^{\nu+1}}$$

From (1) we get similarly

(9) $$\frac{dx+idy}{du}$$

$$=f(u)=Ce^{cu}\frac{\Pi\sigma(u-b)}{\sigma(u-a_1)^{r_1+1}\sigma(u-a_2)^{r_2+1}\ \dots\ \sigma(u-a_m)^{r_m+1}}$$

$\therefore$ (10) $$\frac{dx-idy}{du}$$

$$=\Gamma e^{\gamma u}\frac{\Pi\sigma(u-\beta)}{\sigma(u-\alpha_1)^{r_1+1}\sigma(u-\alpha_2)^{r_2+1}\ \dots\ \sigma(u-\alpha_m)^{r_m+1}}$$

where $\Gamma, \gamma, \beta, \alpha_1, \alpha_2, \ldots \alpha_m$ are the conjugate complex quantities of $C, c, b, a_1, a_2, \ldots a_m$ respectively.

If the argument u (which is an elliptic integral of the first kind, *vide* p. 25) represent an arc of the curves, then

$$dx^2 + dy^2 = du^2 \text{ or } \frac{dx + i\,dy}{du} \cdot \frac{dx - i\,dy}{du} = 1.$$

Therefore from (9) and (10) it follows that

$$(11) \begin{cases} C\Gamma = 1, \; c + \gamma = 0 \\ \Pi\sigma(u - b) = \sigma(u - a_1)^{r_1 + 1}\sigma(u - a_2)^{r_2 + 1} \ldots \sigma(u - a_m)^{r_m + 1} \end{cases}$$

are the sufficient and necessary conditions that the curves (7) expressed by the integral $\int f(u)du$ be algebraic.

These conditions are satisfied if (since $b_1, b_2 \ldots$ are the zeros of $f(u)$)

$$(12) \begin{cases} f(a_1) = 0 & f'(a_1) = 0 & \ldots & f^{(r_1)}(a_1) = 0 \\ f(a_2) = 0 & f'(a_2) = 0 & \ldots & f^{(r_2)}(a_2) = 0 \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ f(a_m) = 0 & f'(a_m) = 0 & \ldots & f^{(r_m)}(a_m) = 0. \end{cases}$$

We have here $m + r_1 + r_2 + \ldots + r_m$ or $m + r$ equations of condition, which can be satisfied since we have the m quantities $a_1, a_2 \ldots a_m$, and the r coefficients c at our disposal. From the former we get as in Serret's case the modulus of the elliptic function. In trying to find the modulus there are in practice very great, if not sometimes insuperable, algebraic difficulties; but we can generally simplify the process, for $x + iy = \int f(u)du$ will frequently lead to an expression containing various σ – functions and their logarithmic derivatives, and from the properties of such functions we can get many of the conditions to disappear, making it easier to find quantities to satisfy the remaining conditions. We shall illustrate this by three examples.

First Example.

(13) Let $\dfrac{dx + i\,dy}{du} = f(u)$

$$= \sum_{\nu=1}^{m} c_{1,\nu} \frac{d^{2\nu+1}\log\sigma(u - a_1)}{du^{2\nu+1}} + \sum_{\nu=1}^{n} c_{2,\nu} \frac{d^{2\nu+1}\log\sigma(u - a_2)}{du^{2\nu+1}}$$

with the further condition that a_1, a_2 and their conjugates α_1, α_2 are such that their differences $\alpha_1 - a_1$, $\alpha_1 - a_2$, $\alpha_2 - a_1$, $\alpha_2 - a_2$ are semiperiods.

Comparing (13) with (8) we see that the conditions corresponding to (12) for the curves being algebraic are

$$(14) \quad \begin{cases} f(\alpha_1) = 0, \;\; f'(\alpha_1) = 0, \;\; f''(\alpha_1) = 0 \;\; \ldots \;\; f^{(2m)}(\alpha_1) = 0 \\ f(\alpha_2) = 0, \;\; f'(\alpha_2) = 0, \;\; f''(\alpha_2) = 0 \;\; \ldots \;\; f^{(2n)}(\alpha_2) = 0. \end{cases}$$

But, by the properties of the σ functions, if λ be a positive integer, then* $\dfrac{d^{2\lambda+1}}{du^{2\lambda+1}}\log\sigma u = 0$, if $u = \omega$, ω', or ω'' $(\omega'' = \omega + \omega')$.

Hence it is evident from (13) that $f(\alpha_1)$, $f(\alpha_2)$ and all their even derivatives vanish, so that in (14) only the following $m + n$ conditions remain to be satisfied :—

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* *cf.* Halphen's F. E., Vol. I., p. 27.

$$\begin{aligned}
\wp'^2 &= 4\wp^3 - g_2\wp - g_3 \text{ (by definition)} \\
\therefore \wp'' &= 6\wp^2 - \tfrac{1}{2}g_2 \\
\wp''' &= 12\wp\wp' = 6(\wp\wp' + \wp\wp') \\
\wp^{\text{IV}} &= 6(\wp\wp'' + 2\wp'\wp' + \wp''\wp) \\
\wp^{\text{V}} &= 6(\wp\wp''' + 3\wp'\wp'' + 3\wp''\wp' + \wp'''\wp) \\
&\ldots\ldots\ldots\ldots \\
\wp^{(2\lambda-1)} &= 6[\wp\wp^{(2\lambda-3)} + (2\lambda - 3)\wp'\wp^{(2\lambda-4)} \\
&+ \frac{(2\lambda-3)(2\lambda-4)}{2!}\wp''\wp^{(2\lambda-5)} + \ldots + (2\lambda-3)\wp^{(2\lambda-4)}\wp' + \wp^{(2\lambda-3)}\wp].
\end{aligned}$$

From this it can be seen that since $\wp'\omega = 0$, $\wp'\omega' = 0$, $\wp'\omega'' = 0$ (see *Formeln*, p. 11) $\therefore$ $\wp'''u$, $\wp^{\text{V}}u$, and all the other odd derivatives of $\wp u$ vanish for $u = \omega$, ω', or ω''. But

$$\frac{d^2\log\sigma u}{du^2} = -\wp u \quad \therefore \quad \frac{d^{2\lambda+1}\log\sigma u}{du^{2\lambda+1}} = -\wp^{(2\lambda-1)}u = 0$$

when $u = \omega$, ω', or ω''.

$$(15)\qquad \begin{cases} f'(a_1)=0, & f'''(a_1)=0\ \ldots\ldots f^{(2m-1)}(a_1)=0 \\ f'(a_2)=0, & f'''(a_2)=0\ \ldots\ldots f^{(2n-1)}(a_2)=0 \end{cases}$$

which are linear and homogeneous equations among the $m+n$ coefficients c, whence the conditions that the curves be algebraic can be satisfied.

We may satisfy the given condition that a_1-a_1, etc., be semi-periods in two ways, viz.,

$$\left.\begin{aligned} a_1&=\frac{\omega}{2}-\frac{\omega'}{2},\quad a_2=-\frac{\omega}{2}+\frac{\omega'}{2} \\ a_1&=\frac{\omega}{2}+\frac{\omega'}{2},\quad a_2=-\frac{\omega}{2}-\frac{\omega'}{2} \end{aligned}\right\}\ \text{or}\ \left\{\begin{aligned} a_1&=-\frac{\omega'}{2},\quad a_2=-\omega+\frac{\omega'}{2} \\ a_1&=\ \ \frac{\omega'}{2},\quad a_2=-\omega-\frac{\omega'}{2}\text{*} \end{aligned}\right.$$

so that from (13) we get two functions $f(u)$ to satisfy the case we are discussing, viz.,

$$(16)\quad f(u)=\sum_{\nu=1}^{m} c_{1,\nu}\frac{d^{2\nu+1}}{du^{2\nu+1}}\log\sigma\left(u-\frac{\omega}{2}+\frac{\omega'}{2}\right) + \sum_{\nu=1}^{n} c_{2,\nu}\frac{d^{2\nu+1}}{du^{2\nu+1}}\log\sigma\left(u+\frac{\omega}{2}-\frac{\omega'}{2}\right)\text{†}$$

and

$$f(u)=\sum_{\nu=1}^{m} c_{1,\nu}\frac{d^{2\nu+1}}{du^{2\nu+1}}\log\sigma\left(u+\frac{\omega'}{2}\right) + \sum_{\nu=1}^{n} c_{2,\nu}\frac{d^{2\nu+1}}{du^{2\nu+1}}\log\sigma\left(u+\omega-\frac{\omega'}{2}\right)$$

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* We get other two sets of values that satisfy the given conditions by changing the sign of $\frac{\omega'}{2}$ in each of the above, but of course these values would just be the conjugates of the foregoing.

† A little consideration will show that the curves represented by this equation (which is erroneously stated by Kiepert on page 11 of his dissertation) *embrace all Serret's curves* $\frac{dx+i\,dy}{dz}=\frac{(z-a)^m(z+a)^n}{(z-a)^{m+1}(z+a)^{n+1}}$. For in the latter we see that there are two infinities which are the same but of opposite signs, viz., $z=+a$, and $z=-a$. So also in the curves represented by (16), remembering that $\frac{d^2\log\sigma u}{du^2}=-\wp u$, and that $\wp(0)=\infty$, and $\wp'(0)=\infty$, we see that the infinities are $u=\frac{\omega}{2}-\frac{\omega'}{2}$ and $u=-\frac{\omega}{2}+\frac{\omega'}{2}$, *i.e.*, they differ only in sign.

SECOND EXAMPLE.

Next take an example suggested by the two equations just given.

(17) Let $\dfrac{dx+i\,dy}{du}=f(u)$

$$=\sum_{\nu=1}^{m} c_{1,\,\nu}\frac{d^{2\nu}}{du^{2\nu}}\left\{\log\sigma\left(u+\frac{\omega'}{2}\right)-\log\sigma\left(u-\omega+\frac{\omega'}{2}\right)\right\}$$

$$+\sum_{\nu=1}^{n} c_{2,\,\nu}\frac{d^{2\nu+1}}{du^{2\nu+1}}\left\{\log\sigma\left(u+\frac{\omega}{2}-\frac{\omega'}{2}\right)-\log\sigma\left(u-\frac{\omega}{2}-\frac{\omega'}{2}\right)\right\}.$$

Comparing this with the general equation (8) we see that

$$\left\{\begin{array}{ll} r_1=r_2=2m-1, & r_3=r_4=2n \\ a_1=-\dfrac{\omega'}{2},\ a_2=\omega-\dfrac{\omega'}{2} & a_3=-\dfrac{\omega}{2}+\dfrac{\omega'}{2},\ a_4=\dfrac{\omega}{2}+\dfrac{\omega'}{2} \\ \alpha_1=+\dfrac{\omega'}{2},\ \alpha_2=\omega+\dfrac{\omega'}{2} & \alpha_3=-\dfrac{\omega}{2}-\dfrac{\omega'}{2},\ \alpha_4=\dfrac{\omega}{2}-\dfrac{\omega'}{2}. \end{array}\right.$$

For the present case the equations of condition corresponding to (12) are

$$(18)\quad\left\{\begin{array}{llcll} f(a_1)=0 & f'(a_1)=0 & \cdot\ \cdot & f^{(2m-2)}(a_1)=0 & f^{(2m-1)}(a_1)=0 \\ f(a_2)=0 & f'(a_2)=0 & \cdot\ \cdot & f^{(2m-2)}(a_2)=0 & f^{(2m-1)}(a_2)=0 \\ f(a_3)=0 & f'(a_3)=0 & \cdot\ \cdot & f^{(2n-1)}(a_3)=0 & f^{(2n)}(a_3)\ \ =0 \\ f(a_4)=0 & f'(a_4)=0 & \cdot\ \cdot & f^{(2n-1)}(a_4)=0 & f^{(2n)}(a_4)\ \ =0 \end{array}\right.$$

If the argument u is increased by ω, then

(19)

$$\log\sigma\left(u+\frac{\omega'}{2}\right)-\log\sigma\left(u-\omega+\frac{\omega'}{2}\right)+\log\sigma\left(u+\frac{\omega}{2}-\frac{\omega'}{2}\right)-\log\sigma\left(u-\frac{\omega}{2}-\frac{\omega'}{2}\right)$$

becomes

$$\log\sigma\left(u+\omega+\frac{\omega'}{2}\right)-\log\sigma\left(u+\frac{\omega'}{2}\right)+\log\sigma\left(u+\frac{3\omega}{2}-\frac{\omega'}{2}\right)-\log\sigma\left(u+\frac{\omega}{2}-\frac{\omega'}{2}\right).$$

But, since $\pm 2\omega$ is one of the periods of the function, this is equal to

$$\log\sigma\left(u-\omega+\frac{\omega'}{2}\right)-\log\sigma\left(u+\frac{\omega'}{2}\right)+\log\sigma\left(u-\frac{\omega}{2}-\frac{\omega'}{2}\right)-\log\sigma\left(u+\frac{\omega}{2}-\frac{\omega'}{2}\right)$$

which is just (19) with a negative sign.

But $\qquad a_2 = a_1 + \omega, \qquad$ and $\quad a_4 = a_3 + \omega$

$\therefore \qquad f(a_2) = -f(a_1) \quad$ and $\quad f(a_4) = -f(a_3)$

so that if $f(a_1)$ vanishes so must $f(a_2)$, and if $f(a_3)$ vanishes so must $f(a_4)$, and the same is true of their derivatives, so that in (18) we may omit the 2nd and 4th rows, and thus the number of necessary conditions is reduced by one-half.

Again, both parts of the right-hand side of (17) belong to the type

$$\log\sigma u - \log\sigma(u-\omega).$$

But by the properties of the $\sigma -$ functions

(20) $\dfrac{d^{2\lambda+1}}{du^{2\lambda+1}}\{\log\sigma u - \log\sigma(u-\omega)\}$ vanishes if $u = \pm\,\omega', \ \pm\,\omega\,\pm\omega'$

and

(21) $\dfrac{d^{2\lambda}}{du^{2\lambda}}\{\log\sigma u - \log\sigma(u-\omega)\}$ vanishes if $u = \pm\dfrac{\omega}{2}, \ \pm\dfrac{\omega}{2}\pm\omega'$.

$$\text{But}\quad \left.\begin{aligned} a_1 - a_1 &= \omega' \\ a_1 - a_2 &= -\omega + \omega' \\ a_1 - a_3 &= \frac{\omega}{2} \\ a_1 - a_4 &= -\frac{\omega}{2} \end{aligned}\right\}$$

$\therefore$ all odd derivatives of $f(a_1)$ vanish [the first half by condition (20), and the second half by (21)].

$$\text{Also}\quad \left.\begin{aligned} a_3 - a_1 &= -\frac{\omega}{2} \\ a_3 - a_2 &= -\frac{3\omega}{2} \\ a_3 - a_3 &= -\omega' \\ a_3 - a_4 &= -\omega - \omega' \end{aligned}\right\}$$

$\therefore$ $f(a_3)$ and all its even derivatives vanish [the first half by (21), and the second half by (20)].

Hence, finally, taking all these into account, (18) reduces to only $m+n$ conditions, viz.,

$$\left\{\begin{array}{lllll} f(a_1)=0 & f''(a_1)=0 & \ldots\ldots & f^{(2m-2)}(a_1)=0 \\ f'(a_3)=0 & f'''(a_3)=0 & \ldots\ldots & f^{(2n-1)}(a_3)=0 \end{array}\right.$$

which can be satisfied since we have the $m+n$ coefficients c at our disposal.

Kiepert works out at great length the particular case of the group of curves (17) when $m = n = 1$.

The equation he finally gets is

$$54(x^4 - y^4) = 36(x^2 + y^2)^3 + 18(x^2 + y^2) - \sqrt{3}$$

whose shape is given in Figure 14.

Third Example.

As another group of the infinity of curves discovered by himself Kiepert takes

$$(22)\quad \frac{dx + i\,dy}{du} = \sum_{\lambda=0}^{n_1} c_{1,\lambda}\,\psi^{(\lambda)}(u + \beta_1 i) + \sum_{\lambda=0}^{n_2} c_{1,\lambda}\,\psi^{(\lambda)}(u + \beta_2 i) + \cdots$$

$$\cdots + \sum_{\lambda=0}^{n_k} c_{1,\lambda}\,\psi^{(\lambda)}(u + \beta_k i)$$

where $\beta_1, \beta_2, \ldots \beta_k$ are real quantities, and for brevity $\psi^{(\lambda)}(u)$ represents

$$\frac{d^{(\lambda+2)}}{du^{(\lambda+2)}}\{\log\sigma u + \epsilon\log\sigma(u + \frac{1}{r}2\hat{\omega}) + \epsilon^2\log\sigma(u + \frac{2}{r}2\hat{\omega}) + \cdots$$

$$\cdots + \epsilon^{r-1}\log\sigma(u + \frac{r-1}{r}2\hat{\omega})\}$$

$$(\lambda = 0, 1, 2, \ldots)$$

where $2\hat{\omega}$ is any period of the elliptic function, and ϵ is the r^{th} root of unity.

He shows that the necessary conditions can be satisfied that (22) may be algebraic.

He examines in detail the case where $r = 3$, $n_1 = 0$, $k = 1$, and $2\hat{\omega} = -4\omega + 2\omega'$.

$$\therefore\ \frac{dx + i\,dy}{du} = \psi\left(u + \frac{2\omega}{3} - \frac{2\omega'}{3}\right)$$

$$= -\frac{d^2}{du^2}\left[\log\sigma\left(u + \frac{2\omega}{3} - \frac{2\omega'}{3}\right) + \epsilon\log\sigma\left(u + \frac{2\omega}{3} - \frac{2\omega'}{3} + \tfrac{1}{3}(-4\omega + 2\omega')\right)\right.$$

$$\left. + \epsilon^2\log\sigma\left(u + \frac{2\omega}{3} - \frac{2\omega'}{3} + \tfrac{2}{3}(-4\omega + 2\omega')\right)\right]$$

$$= \wp\left(u + \frac{2\omega}{3} - \frac{2\omega'}{3}\right) + \epsilon\wp\left(u - \frac{2\omega}{3}\right) + \epsilon^2\wp\left(u + \frac{2\omega'}{3}\right).$$

He shows that the cartesian equation of the curve is
$(x^2+y^2)^2 - y(y^2-3x^2) = 0$ and its polar equation $r^3 = \cos 3\theta$.*

The curve consists of 3 equal parts as in Figure 15. Similarly all the algebraic curves (22) are composed of r equal parts.

IX.

The weakness of both Serret's and Kiepert's methods for plane curves is that they cannot be extended to space curves. As we have seen, they both start from the expression $x+iy$; they form an elliptic function for this, and find the conditions that the curve be algebraic, *i.e.* that $\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 = \textit{constant}$. Of course such a method is not applicable to curves of double curvature. To get a method applicable at once to curves of single and double curvature Kiepert's method has been somewhat modified by R. von Lilienthal in a dissertation entitled *Zur Theorie der Curven deren Bogenlänge ein elliptisches Integral erster Art ist* (Berlin, 1882). He proceeds thus: he puts the co-ordinates x and y (and z also for a space curve) equal to certain elliptic functions. Then he forms the expression for $\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2$ or $\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$ and finds the condition that these sums are constant, *i.e.* that the co-efficients of the infinite terms vanish.

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* Observe the analogy of this curve to the lemniscate $r^2 = \cos 2\theta$, which of course consists of two equal loops.

We can very easily reduce the expression for an arc u of $r^3 = \cos 3\theta$ to an elliptic integral of the first kind; for

$$du = \sqrt{dr^2 + r^2 d\theta^2} = \frac{dr^2}{\sqrt{1-r^6}}$$

Take $r^2 = \frac{1}{\rho}$, $\therefore$ the above becomes

$$du = -\frac{d\rho}{\sqrt{4\rho^3 - 4}} \text{ or } u = -\int \frac{d\rho}{\sqrt{4\rho^3-4}}$$

Kiepert shows in *Crelle's Journal*, Vol. LXXIV. (1872), how to divide an arc of the curve $r^3 = \cos 3\theta$ into 7, 13, 19, 31 or, generally, $6q+1$ equal parts.

Lilienthal illustrates his method by applying it to the case of the plane lemniscate and the "spherical" lemniscate, and in the latter part of his paper he discusses the connection between spherical curves whose arcs are elliptic integrals of the first kind and surfaces of minimum area for given perimeter (*Minimalflächen*).*

Starting with the lemniscate, he puts

$$-x=$$

$$\frac{\mathrm{A}d^2\log\sigma(u-a)}{du^2}+\frac{\mathrm{A}'d^2\log\sigma(u-a')}{du^2}+\frac{\mathrm{A}_1d^2\log\sigma(u-a_1)}{du^2}+\frac{\mathrm{A}_1'd^2\log\sigma(u-a_1')}{du^2}$$

or

$$(1)\quad \begin{cases} x=\mathrm{A}\wp(u-a)+\mathrm{A}'\wp(u-a')+\mathrm{A}_1\wp(u-a_1)+\mathrm{A}_1'\wp(u-a_1') \\ y=\mathrm{B}\wp(u-a)+\mathrm{B}'\wp(u-a')+\mathrm{B}_1\wp(u-a_1)+\mathrm{A}_1'\wp(u-a_1') \end{cases}$$

where a', a_1', A', A_1', B', B_1' are conjugate respectively to a, a_1, A, A_1, B, B_1.

Our problem is to determine those quantities so that $\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2=$ *constant.*

We have

$$(2)\quad \frac{dx}{du}=\mathrm{A}\wp'(u-a)+\mathrm{A}'\wp'(u-a')+\mathrm{A}_1\wp'(u-a_1)+\mathrm{A}_1'\wp'(u-a_1').$$

If we develop this in the neighbourhood of $u=a$ we get †

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* For the properties of *minimalflächen* see Todhunter's *History of the Progress of the Calculus of Variations* (1861); Riemann's Memoir *Ueber die Fläche vom kleinsten Inhalt bie gegebener Begrenzung*, revised by K. Hattendorff, *Bd.* 13 *der Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen* (1867); and particularly H. A. Schwarz's article *Miscellen aus dem Gebiete der Minimalflächen*, at pp. 168-189, vol. i., of his *Mathematische Abhandlungen* (Berlin, 1890).

† For the development of $\wp'(u-a)$ see *Formeln*, p. 11, and for the other three terms we use the theorem that if $f(u)$ be an elliptic function of u, then its development in the neighbourhood of $u=a$ is

$$f(u)=f(a)+(u-a)f'(a)+\frac{(u-a)^2}{2!}f''(a)+\frac{(u-a)^3}{3!}f'''(a)+\ .\ .\ .$$

$$.\ .\ .+\frac{(u-a)^n}{n!}f^{(n)}(a).$$ (*Cf.* Forsyth's *Theory of Functions of a Complex Variable*, p. 50).

$$\frac{dx}{du} = -\frac{2A}{(u-a)^3} + \frac{Ag_2}{10}(u-a) + \dots$$

$$+ A'\,p'(a-a') + (u-a)A'\,p''(a-a') + \frac{(u-a)^2 A'}{2}p'''(a-a') + \dots$$

$$+ A_1\,p'(a-a_1) + (u-a_1)A_1\,p''(a-a_1) + \frac{(u-a_1)^2 A_1}{2}p'''(a-a_1) + \dots$$

$$+ A_1'p'(a-a_1') + (u-a_1')A_1'p''(a-a_1') + \frac{(u-a_1')^2 A_1'}{2}p'''(a-a_1') + \dots$$

$$= -\frac{2A}{(u-a)^3} + A'p'(a-a') + A_1p'(a-a_1) + A_1'p'(a-a_1')$$

$$+ \left\{A'p''(a-a') + A_1p''(a-a_1) + A_1'p''(a-a_1') + \frac{Ag_2}{10}\right\}(u-a)$$

$$+ \left\{A'p'''(a-a') + A_1p'''(a-a_1) + A_1'p'''(a-a_1')\right\}\frac{(u-a)^2}{2} + \dots$$

$$\therefore \left(\frac{dx}{du}\right)^2 = \frac{4A^2}{(u-a)^6} - \frac{4A\{A'p'(a-a') + A_1p'(a-a_1) + A_1'p'(a-a_1')\}}{(u-a)^3}$$

$$-\frac{4A\left\{A'p''(a-a') + A_1p''(a-a_1) + A_1'p''(a-a_1') + \frac{Ag_2}{10}\right\}}{(u-a)^2}$$

$$-\frac{2A\{A'p'''(a-a') + A_1p'''(a-a_1) + A_1'p'''(a-a_1')\}}{u-a} + P(u-a),$$

where $P(u-a)$ is finite at the point $u=a$.

Similarly, in the neighbourhood of $u=a_1$ we get

$$\left(\frac{dx}{du}\right)^2 = \frac{4A_1^2}{(u-a_1)^6} - \frac{4A_1\{Ap'(a_1-a) + A'p'(a_1-a') + A_1'p'(a_1-a_1')\}}{(u-a_1)^3}$$

$$-\frac{4A_1\left\{Ap''(a_1-a) + A'p''(a_1-a') + A_1'p''(a_1-a_1') + \frac{A_1g_2}{10}\right\}}{(u-a_1)^2}$$

$$-\frac{2A_1\{Ap'''(a_1-a) + A'p'''(a_1-a') + A'_1p'''(a_1-a_1')\}}{u-a_1} + P_1(u-a_1).$$

By putting B for A throughout, we get the corresponding two expressions for $\left(\frac{dy}{du}\right)^2$.

Now, in order that $\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 =$ *constant*, it is evidently necessary that the infinite terms—*i.e.*, those having $u-a$ and $u-a_1$ in the denominator, vanish. The conditions for this are

$$\text{I.}\quad \begin{cases} A^2 + B^2 = 0 \\ A_1^2 + B_1^2 = 0 \end{cases}$$

$$\text{II.}\quad \begin{cases} (AA' + BB')\wp'(a - a') + (AA_1 + BB_1)\wp'(a - a_1) + (AA_1' + BB_1')\wp'(a - a_1') = 0 \\ (AA_1 + BB_1)\wp'(a_1 - a) + (A'A_1 + B'B_1)\wp'(a_1 - a') + (A_1A_1' + B_1B_1')\wp'(a_1 - a_1') = 0 \end{cases}$$

$$\text{III.}\quad \begin{cases} (AA' + BB')\wp''(a - a') + (AA_1 + BB_1)\wp''(a - a_1) + (AA_1' + BB_1')\wp''(a - a_1') = 0 \\ (AA_1 + BB_1)\wp''(a_1 - a) + (A'A_1 + B'B_1)\wp''(a_1 - a') + (A_1A_1' + B_1B_1')\wp''(a_1 - a_1') = 0 \end{cases}$$

$$\text{IV.}\quad \begin{cases} (AA' + BB')\wp'''(a - a') + (AA_1 + BB_1)\wp'''(a - a_1) + (AA_1' + BB_1')\wp'''(a - a_1') = 0 \\ (AA_1 + BB_1)\wp'''(a_1 - a) + (A'A_1 + B'B_1)\wp'''(a_1 - a') + (A_1A_1' + B_1B_1')\wp'''(a_1 - a_1') = 0. \end{cases}$$

From I. we get $B = \pm iA$, $B_1 = \pm iA_1$. In each case we take the $+$ sign since we suppose x and y to have the same sign.

Hence we have

$$(3) \qquad \begin{cases} B = iA\,, & B' = -iA' \\ B_1 = iA_1, & B_1' = -iA_1' \end{cases}$$

$$\begin{aligned} \therefore \quad & A\,A' + B\,B' = 2A\,A' \\ & A\,A_1 + B\,B_1 = 0 \\ & A\,A_1' + B\,B_1' = 2A\,A_1' \\ & A'A_1 + B'B_1 = 2A'A_1 \\ & A_1A_1' + B_1B_1' = 2A_1A_1' \end{aligned}$$

so that the foregoing equations of condition reduce to

$$\left.\begin{aligned} A'\mathrm{p}'(a-a') + A_1'\mathrm{p}'(a-a_1') = 0 \\ A'\mathrm{p}'(a_1-a') + A_1'\mathrm{p}'(a_1-a_1') = 0 \end{aligned}\right\}\text{from II.}$$

$$\left.\begin{aligned} A'\mathrm{p}''(a-a') + A_1'\mathrm{p}''(a-a_1') = 0 \\ A'\mathrm{p}''(a_1-a') + A_1'\mathrm{p}''(a_1-a_1') = 0 \end{aligned}\right\}\text{from III.}$$

$$\left.\begin{aligned} A'\mathrm{p}'''(a-a') + A_1'\mathrm{p}'''(a-a_1') = 0 \\ A'\mathrm{p}'''(a_1-a') + A_1'\mathrm{p}'''(a_1-a_1') = 0 \end{aligned}\right\}\text{from IV.}$$

Now these conditions will evidently be fulfilled if we can find an elliptic function which will be zero at a and a_1, and infinite at a' and a'_1. Hence the function has the form

$$f(u) = c\mathrm{p}'(u-a') + c_1\mathrm{p}'(u-a_1').$$

Also, we see from the above equations of condition that the zeros and infinities are each of the 3rd order, so that we may also represent $f(u)$ as in the equation viii. (1).

Therefore we may write

$$(4) \qquad \frac{1}{C}f(u) = \frac{\sigma^3(u-a)\sigma^3(u-a_1)}{\sigma^3(u-a')\sigma^3(u-a_1')} = c\mathrm{p}'(u-a') + c_1\mathrm{p}'(u-a_1')$$

and these are the equations we have to satisfy.

If we develop $f(u)$ in the neighbourhood of a' we get,* remembering that

$$\frac{d}{du}\log\sigma(u-a) = \frac{\sigma'}{\sigma}(u-a)\dagger, \text{ and } \frac{d^2}{du^2}\log\sigma(u-a) = -\mathrm{p}(u-a),$$

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* *Formeln*, p. 10, and footnote p. 26.

† It ought perhaps to have been mentioned that

$$\frac{\sigma'}{\sigma}u \text{ is a contraction for } \frac{\sigma' u}{\sigma u}.$$

$$(5)\ \frac{1}{C}f(u)=\frac{\sigma^3(a'-a)\sigma^3(a'-a_1)}{\sigma^3(a'-a_1')}\times$$

$$\Bigg[\frac{1}{(u-a')^3}+3\left\{\frac{\sigma'}{\sigma}(a'-a)+\frac{\sigma'}{\sigma}(a'-a_1)-\frac{\sigma'}{\sigma}(a'-a_1')\right\}\frac{1}{(u-a')^2}$$

$$+\Bigg\{9\left(\frac{\sigma'}{\sigma}(a'-a)+\frac{\sigma'}{\sigma}(a'-a_1)-\frac{\sigma'}{\sigma}(a'-a_1')\right)^2$$

$$-3\Big(\mathrm{p}(a'-a)+\mathrm{p}(a'-a_1)-\mathrm{p}(a'-a_1')\Big)\Bigg\}\frac{1}{2(u-a')}+\ \ldots\Bigg]$$

By putting a_1' instead of a' we get the corresponding expression or the development of the function in the neighbourhood of a_1.

For shortness put

$$(6)\ \begin{cases}\mathrm{K}\ \text{ for }\ \dfrac{\sigma^3(a'-a)\sigma^3(a'-a_1)}{\sigma^3(a'-a_1')}\\[2ex] \mathrm{K}_1\ \text{ for }\ \dfrac{\sigma'}{\sigma}(a'-a)+\dfrac{\sigma'}{\sigma}(a'-a_1)-\dfrac{\sigma'}{\sigma}(a'-a_1')\\[2ex] \mathrm{K}_2\ \text{ for }\ \mathrm{p}(a'-a)+\mathrm{p}(a'-a_1)-\mathrm{p}(a'-a_1')\end{cases}$$

and let K', K_1', K_2' be the corresponding expressions when we put a_1 for a, and a_1' for a'.

Remembering also that

$$*\ \ \mathrm{p}'(u-a')=-\frac{2}{(u-a')^3}+\ \ldots\ldots$$

$$\dagger\ \ \mathrm{p}(u-a')=\frac{1}{(u-a')^2}+\ \ldots\ldots$$

$$\dagger\ \ \frac{\sigma'}{\sigma}(u-a')=\frac{1}{u-a'}+\ \ldots\ldots$$

and that $\dfrac{\sigma'}{\sigma}(-u)=-\dfrac{\sigma'}{\sigma}(u)$, and $\mathrm{p}(-u)=\mathrm{p}u$.

we get from (4) and (5),

$$(7)\ \frac{1}{C}f(u)=\frac{\sigma^3(u-a)\sigma^3(u-a_1)}{\sigma^3(u-a')\sigma^3(u-a_1')}=-\frac{\mathrm{K}}{2}\mathrm{p}'(u-a')-\frac{\mathrm{K}'}{2}\mathrm{p}'(u-a_1')$$

$$+3\mathrm{K}\mathrm{K}_1\mathrm{p}(u-a')+3\mathrm{K}'\mathrm{K}_1'\mathrm{p}(u-a_1')+\tfrac{1}{2}\mathrm{K}(9\mathrm{K}_1^2-3\mathrm{K}_2)\frac{\sigma'}{\sigma}(u-a')$$

$$+\tfrac{1}{2}\mathrm{K}'(9\mathrm{K}_1'^2-3\mathrm{K}_2')\frac{\sigma'}{\sigma}(u-a_1')+\textit{constant.}$$

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* *Formeln*, p. 11.

† *Formeln*, p. 10.

By putting $u=0$ on both sides we find that the

$$(8)\ \textit{Constant} = \frac{\sigma^3(a)\sigma^3(a_1)}{\sigma^3(a')\sigma^3(a_1')} - \frac{K}{2}p'(a') - \frac{K'}{2}p'(a_1') - 3KK_1p(a')$$

$$- 3K'K_1'p(a_1') + \frac{K}{2}(9K_1^2 - 3K_2)\frac{\sigma'}{\sigma}(a') + \frac{K'}{2}(9K_1'^2 - 3K_2')\frac{\sigma'}{\sigma}(a_1').$$

We see from (7) that the necessary and sufficient conditions that $f(u)$ take the second form required in (4) are that K_1, K_1', K_2, K_2' as well as the constant term vanish, for then (7) will become

$$(7a)\ \frac{1}{C}f(u) = \frac{\sigma^3(u-a)\sigma^3(u-a_1)}{\sigma^3(u-a')\sigma^3(u-a_1')} = -\frac{K}{2}p'(u-a') - \frac{K'}{2}p'(u-a_1').$$

But the sum of the zeros equals the sum of the infinities [see VIII. (2)].

$$\left.\begin{array}{rl} i.e.\ a + a_1 = & a' + a_1' \\ \therefore\ a' - a = & -(a_1' - a_1) \\ \text{and } a' - a_1 = & -(a_1' - a\ ) \\ \text{and of course } a' - a_1' = & -(a_1' - a'\ ) \end{array}\right\} \begin{array}{l} \therefore \text{ from (6)} \\ K = -K',\ K_1 = -K_1',\ K_2 = K_2'. \end{array}$$

Wherefore the conditions that K_1, K_1', K_2, K_2', and the *constant* in (7) vanish are equivalent as we see from (6) to finding the conditions that

$$(9)\qquad \frac{\sigma'}{\sigma}(a'-a) + \frac{\sigma'}{\sigma}(a'-a_1) - \frac{\sigma'}{\sigma}(a'-a_1') = 0$$

$$\text{and } (10)\qquad p\,(a'-a) + p\,(a'-a_1) - p\,(a'-a_1') = 0$$

and also *constant* $=0$.

But since $a'-a_1' = (a'-a) + (a'-a_1)$; then applying the two addition formulae.

$$\frac{\sigma'}{\sigma}(u+v) = \frac{\sigma'}{\sigma}(u) + \frac{\sigma'}{\sigma}(v) + \tfrac{1}{2}\frac{p'u - p'v}{pu - pv}\ *$$

$$p(u+v) = \tfrac{1}{4}\left(\frac{p'u - p'v}{p\,u - p\,v}\right)^2 - pu - pv\ \dagger$$

* *Formeln*, p. 13.

† *Formeln*, p. 14.

(9) and (10) are reduced respectively to finding the conditions that

$$(11)\qquad \frac{\wp'(a'-a)-\wp'(a'-a_1)}{\wp(a'-a)-\wp(a'-a_1)}=0$$

$$(12)\qquad \wp(a'-a)+\wp(a'-a_1)=0.$$

But, as already stated,

$$\wp'\omega=0,\quad \wp'\omega'=0,\quad \wp'\omega''=0$$

$$\wp\,\omega=e_1,\quad \wp\,\omega'=e_3,\quad \wp\,\omega''=e_2.$$

We see, therefore, that (11) will be satisfied if $a'-a$, $a'-a_1$ are semiperiods; and, since $e_1+e_2+e_3=0$ (see p. 26), we see that (12) will also be satisfied if one of the e's vanishes. To find which e can vanish, we proceed thus:

k, the modulus of the elliptic function, must be $+^{ve}$ and <1, and

$$k^2=\frac{e_2-e_3}{e_1-e_3}.\,*$$

If $e_1=0$, then $k^2=2$; if $e_2=0$, then $k^2=\frac{1}{2}$; if $e_3=0$, then $k^2=-1$; so that we can only possibly have $e_2=0$.

$\therefore$ (13)

$$a'-a=\omega';\ a'-a_1=\omega;\ a'-a'_1=(a'-a_1)+(a'-a)=\omega+\omega'=\omega''.$$

Putting these values in (6) we get

$$K=\frac{\sigma^3(\omega')\,\sigma^3(\omega)}{\sigma^3(\omega'')};\ \text{and, as we have already seen, } K'=-K.$$

With these data it can now be shown that the *constant* (8) vanishes, and therefore the proof is complete that the necessary conditions can be fulfilled that the third member of (7) is reducible to the form of the third member of (4), which in turn we have shown to be the condition that the curve is algebraic.

To find the cartesian equation of the curve we may proceed thus:

Insert the above values of K and K′ in (7a), and at the same time put

$$\frac{1}{C}=-\frac{\sigma^3(\omega)\,\sigma^3(\omega')}{2\sigma^3(\omega'')};$$

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* *Formeln*, p. 30.

we then get

$$f(u) = p'(u-a') - p'(u-a_1').$$

Comparing this with our original equations (1) and (2) we see that

$$A' = 1, \qquad A_1' = -1.$$

Also, according to (13) we can put *

$$(14) \qquad \begin{cases} a = \frac{\omega}{2} - \frac{\omega'}{2}, & a_1 = -\frac{\omega}{2} + \frac{\omega'}{2} \\ a' = \frac{\omega}{2} + \frac{\omega'}{2}, & a_1' = -\frac{\omega}{2} - \frac{\omega'}{2}. \end{cases}$$

Hence in (1) we have, remembering also the values in (3),

$$x = p\left(u - \frac{\omega}{2} + \frac{\omega'}{2}\right) + p\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right) - p\left(u + \frac{\omega}{2} - \frac{\omega'}{2}\right) - p\left(u + \frac{\omega}{2} + \frac{\omega'}{2}\right)$$

$$y = ip\left(u - \frac{\omega}{2} + \frac{\omega'}{2}\right) - ip\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right) - ip\left(u + \frac{\omega}{2} - \frac{\omega'}{2}\right) + ip\left(u + \frac{\omega}{2} + \frac{\omega'}{2}\right).$$

From this we get

$$(15) \qquad x + iy = 2\left[p\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right) - p\left(u + \frac{\omega}{2} + \frac{\omega'}{2}\right)\right]$$

$$(16) \qquad x - iy = 2\left[p\left(u - \frac{\omega}{2} + \frac{\omega'}{2}\right) - p\left(u + \frac{\omega}{2} - \frac{\omega'}{2}\right)\right]$$

But $p\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right) = p\left(u - \frac{\omega''}{2}\right)$ and $p\left(u + \frac{\omega}{2} + \frac{\omega'}{2}\right) = p\left(u + \frac{\omega''}{2}\right)$

Also (see *Formeln*, p. 14).

$$(17) \quad p(u+v) \,.\, p(u-v) = \frac{(pu pv + \frac{1}{4}g_2)^2 + g_3(pu + pv)}{(pu - pv)^2}$$

Remembering that $e_2 = 0$

$$\left.\begin{aligned} \text{and that } \tfrac{1}{4}g_2 &= -(e_2e_3 + e_3e_1 + e_1e_2) \\ \tfrac{1}{4}g_3 &= \quad e_1e_2e_3 \end{aligned}\right\} \textit{(Formeln, p. 12)}$$

$\therefore$ in this case $\quad \frac{1}{4}g_2 = e_3^2$, and $g_3 = 0$.

Wherefore (17) gives us

$$(18) \qquad p\left(u + \frac{\omega''}{2}\right) \,.\, p\left(u - \frac{\omega''}{2}\right) = \frac{\left(pu\, p\frac{\omega''}{2} + e_3^2\right)^2}{\left(pu - p\frac{\omega''}{2}\right)^2}$$

* Compare these values with Kiepert's values on p. 32 for the group of curves which includes the lemniscate.

Again (see *Formeln*, p. 14)

$$\frac{(\mathrm{p}^2u + \tfrac{1}{4}g_2)^2 + 2g_3\mathrm{p}u}{4\mathrm{p}^3u - g_2\mathrm{p}u - g_3} = \mathrm{p}(2u)$$

Therefore,
$$\frac{\left\{\mathrm{p}^2\left(\frac{\omega''}{2}\right) + e_3^2\right\}^2}{\text{etc.}} = \mathrm{p}\omega'' = e_2 = 0$$

whence
$$\mathrm{p}\frac{\omega''}{2} = ie_3\,.$$

Therefore (18) gives

$$\mathrm{p}\left(u + \frac{\omega''}{2}\right) \cdot \mathrm{p}\left(u - \frac{\omega''}{2}\right) = \left(\frac{\mathrm{p}u \,.\, ie_3 + e_3^2}{\mathrm{p}u - ie_3}\right)^2 = -e_3^2$$

$$\therefore\ \mathrm{p}\left(u + \frac{\omega''}{2}\right) \quad \text{or} \quad \mathrm{p}\left(u + \frac{\omega}{2} + \frac{\omega'}{2}\right) = \frac{-e_3^2}{\mathrm{p}\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right)}$$

Similarly
$$\mathrm{p}\left(u - \frac{\omega}{2} + \frac{\omega'}{2}\right) = e_3 + \frac{2e_3^2}{\mathrm{p}\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right) - e_3}$$

and
$$\mathrm{p}\left(u + \frac{\omega}{2} - \frac{\omega'}{2}\right) = e_3 - \frac{2e_3\mathrm{p}\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right)}{e_3 + \mathrm{p}\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right)}$$

Putting these values in (15) and (16) they reduce to

$$x + iy = 2\,\frac{\mathrm{p}^2\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right) + e_3^2}{\mathrm{p}\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right)}$$

$$x - iy = 4e_3\frac{\mathrm{p}^2\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right) + e_3^2}{\mathrm{p}^2\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right) - e_3^2}.$$

Eliminating the p – function between these, we get finally

$$(x^2 + y^2)^2 = 32e_3^2(x^2 - y^2)$$

i.e., the curve is a lemniscate.

X.

As we have already said, the great advantage of Lilienthal's notation is that it can be readily extended to space curves. To show this Lilienthal takes the case analogous to the plane lemniscate. We have

$$(1)\qquad \begin{cases} x = \mathrm{A}\mathrm{p}(u-a) + \mathrm{A}'\mathrm{p}(u-a') + \mathrm{A}_1\mathrm{p}(u-a_1) + \mathrm{A}_1'\mathrm{p}(u-a_1') \\ y = \mathrm{B}\mathrm{p}(u-a) + \mathrm{B}'\mathrm{p}(u-a') + \mathrm{B}_1\mathrm{p}(u-a_1) + \mathrm{B}_1'\mathrm{p}(u-a_1') \\ z = \mathrm{C}\mathrm{p}(u-a) + \mathrm{C}'\mathrm{p}(u-a') + \mathrm{C}_1\mathrm{p}(u-a_1) + \mathrm{C}_1'\mathrm{p}(u-a_1'). \end{cases}$$

In order that $\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$ have no point at infinity we have (see p. 39)

$$\text{I.}\quad \begin{cases} \mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 = 0 \\ \mathrm{A}_1^2 + \mathrm{B}_1^2 + \mathrm{C}_1^2 = 0 \end{cases}$$

$$\text{II.}\quad \begin{cases} (\mathrm{AA}' + \mathrm{BB}' + \mathrm{CC}')\mathrm{p}'(a-a') + (\mathrm{A}\,\mathrm{A}_1 + \mathrm{B}\,\mathrm{B}_1 + \mathrm{C}\,\mathrm{C}_1)\mathrm{p}'(a-a_1) \\ \qquad\qquad + (\mathrm{A}\,\mathrm{A}_1' + \mathrm{B}\,\mathrm{B}_1' + \mathrm{C}\,\mathrm{C}_1')\mathrm{p}'(a-a_1') = 0 \\ (\mathrm{AA}_1 + \mathrm{BB}_1 + \mathrm{CC}_1)\mathrm{p}'(a_1-a) + (\mathrm{A}'\mathrm{A}_1 + \mathrm{B}'\mathrm{B}_1 + \mathrm{C}'\mathrm{C}_1)\mathrm{p}'(a_1-a') \\ \qquad\qquad + (\mathrm{A}_1'\mathrm{A}_1 + \mathrm{B}_1'\mathrm{B}_1 + \mathrm{C}_1'\mathrm{C}_1)\mathrm{p}'(a_1-a_1') = 0 \end{cases}$$

and systems corresponding to III. and IV. by putting p″ and p‴ instead of p′ in II.

These equations, as we have seen, assume the simplest form when a, a', a_1, and a_1' have the values given in IX. (14); so that

$$\begin{aligned} a - a' &= -(a_1 - a_1') = -\omega' = \omega' \text{ (since } 2\omega' \text{ is a period)} \\ a - a_1 &= \quad \omega - \omega' \;=\; \omega + \omega' = \omega'' \\ a - a_1' &= -(a_1 - a') = \quad \omega. \end{aligned}$$

But $\mathrm{p}'\omega$, $\mathrm{p}'\omega'$, and $\mathrm{p}'\omega''$ are zero, therefore also p‴ vanishes for the semiperiods,* so that the conditions II. and IV. are at once satisfied, and we are left to find the necessary and sufficient conditions that

$$\text{I.}\quad \begin{cases} \mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 = 0 \\ \mathrm{A}_1^2 + \mathrm{B}_1^2 + \mathrm{C}_1^2 = 0 \end{cases}$$

* Halphen's F. E., p. 27.

$$\text{III.}\begin{cases}(AA'+BB'+CC')p''\omega' + (A\,A_1+B\,B_1+C\,C_1)p''\omega'' \\ \qquad\qquad\qquad + (A\,A_1'+B\,B_1'+C\,C_1')p''\omega \;=0 \\ (AA_1+BB_1+CC_1)p''\omega''+(A'A_1+B'B_1+C'C_1)p''\omega \\ \qquad\qquad\qquad +(A_1'A_1+B_1'B_1+C_1'C_1)p''\omega'=0.\end{cases}$$

Let us now put

$$A = r\,e^{i\alpha}, \quad B=\rho\, e^{i\beta}, \quad C=R\,e^{i\gamma}, \quad A'=r\,e^{-i\alpha}, \text{ etc.}$$

$$A_1=r_1e^{i\alpha_1}, \quad B_1=\rho_1e^{i\beta_1}, \quad C_1=R_1e^{i\gamma_1}, \quad A_1'=r_1e^{-i\alpha_1}, \text{ etc.}$$

so that I. becomes, remembering that $e^{2i\alpha}=\cos 2\alpha + i\sin 2\alpha$,

$$\Sigma r^2(\cos 2\alpha + i\sin 2\alpha)=0$$
$$\Sigma r_1^2(\cos 2\alpha_1 + i\sin 2\alpha_1)=0$$

or, separating the real and imaginary parts,

$$\text{I}^a.\begin{cases}\Sigma r^2\cos 2\alpha=0\\ \Sigma r^2\sin 2\alpha=0\\ \Sigma r_1^2\cos 2\alpha_1=0\\ \Sigma r_1^2\sin 2\alpha_1=0.\end{cases}$$

III. becomes

$$p''\omega'(r^2+\rho^2+R^2)+p''\omega''\Sigma rr_1e^{i\alpha+i\alpha_1}+p''\omega\Sigma rr_1e^{i\alpha-i\alpha_1}=0$$

and another similar expression.

But $\qquad e^{i\alpha+i\alpha_1}=(\cos\alpha+i\sin\alpha)(\cos\alpha_1+i\sin\alpha_1)$
$$=\cos(\alpha+\alpha_1)+i\sin(\alpha+\alpha_1)$$

Similarly $\quad e^{i\alpha-i\alpha_1}=\cos(\alpha-\alpha_1)+i\sin(\alpha-\alpha_1).$

Hence, putting these values in the above and separating the real and imaginary parts we get

$$\begin{cases}p''\omega'\,\Sigma r^2 + p''\omega''\Sigma rr_1\cos(\alpha+\alpha_1)+p''\omega\Sigma rr_1\cos(\alpha-\alpha_1)=0\\ p''\omega''\Sigma rr_1\sin(\alpha+\alpha_1)+p''\omega\Sigma rr_1\sin(\alpha-\alpha_1)=0\\ p''\omega'\,\Sigma r_1^2+p''\omega''\Sigma rr_1\cos(\alpha+\alpha_1)+p''\omega\Sigma rr_1\cos(\alpha-\alpha_1)=0\\ p''\omega''\Sigma rr_1\sin(\alpha+\alpha_1)+p''\omega\Sigma rr_1\sin(\alpha_1-\alpha)=0.\end{cases}$$

By comparing these four equations we see that we can replace them by the following:—

$$\text{III}^a. \begin{cases} p''\omega'\Sigma r^2 + p''\omega''\Sigma rr_1\cos(a+a_1) + p''\omega\Sigma rr_1\cos(a-a_1) = 0 \\ \Sigma r^2 = \Sigma r_1^{\ 2} \\ \Sigma rr_1 \sin a \cos a_1 = 0 \\ \Sigma rr_1 \cos a \sin a_1 = 0. \end{cases}$$

I^a. and III^a. contain eight equations of condition among twelve unknowns, but one of the quantities r, ρ, R, r_1, ρ_1, R_1 we can from the beginning assume equal to any quantity we choose, and the three remaining undetermined quantities only refer to the rotating of the original coordinate system, so that we have sufficient data to satisfy the necessary conditions for the existence of a curve of double curvature corresponding to the plane lemniscate.

Lilienthal goes through a long analysis in order to obtain the equation of the spherical lemniscate from the equations of condition I^a. and III^a. He finally obtains

$$x = \text{C}\sin\phi \times$$
$$\left[p\left(u - \frac{\omega}{2} + \frac{\omega'}{2}\right) + p\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right) - p\left(u + \frac{\omega}{2} - \frac{\omega'}{2}\right) - p\left(u + \frac{\omega}{2} + \frac{\omega'}{2}\right)\right]$$

$$y = \text{C}\cos\phi \times$$
$$\left[p\left(u - \frac{\omega}{2} + \frac{\omega'}{2}\right) + p\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right) + p\left(u + \frac{\omega}{2} - \frac{\omega'}{2}\right) + p\left(u + \frac{\omega}{2} + \frac{\omega'}{2}\right)\right]$$

$$z = i\text{C} \times$$
$$\left[p\left(u - \frac{\omega}{2} + \frac{\omega'}{2}\right) - p\left(u - \frac{\omega}{2} - \frac{\omega'}{2}\right) - p\left(u + \frac{\omega}{2} - \frac{\omega'}{2}\right) + p\left(u + \frac{\omega}{2} + \frac{\omega'}{2}\right)\right]$$

where C is a constant, and $\sin\phi = \frac{k^2}{1-k^2}$, k being the modulus of the elliptic function.

He proves that this curve lies on the sphere *

* Lilienthal does not remark that if $\phi = \frac{\pi}{2}$, then the radius of this sphere is infinite and the above curve becomes a plane lemniscate, for its modulus is $\frac{1}{\sqrt{2}}$ $\left(\text{since } \sin\phi = \frac{k^2}{1-k^2}\right)$, and the above equations for x and z (y of course vanishes) are the same as those for the plane lemniscate on p. 44. See also first footnote on p. 14.

$$x^2 + \left(y - 2C\frac{e_3\sin^2\phi - e_1}{\cos\phi}\right)^2 + z^2 = \frac{4C^2\{e_2 - e_3 + (e_1 - e_2)\sin^2\phi\}^2}{\cos^2\phi},$$

and on the hyperbolic cylinder

$$\frac{(y + 2e_3C\cos\phi)^2}{4C^2\cos^2\phi(e_1 - e_2)^2} - \frac{x^2}{4C^2\sin^2\phi(e_1 - e_2)^2} = 1,$$

and on the elliptic cylinder

$$\frac{z^2}{4C^2(e_2 - e_3)^2} + \frac{(y + 2e_1C\cos\phi)^2}{4C^2(e_2 - e_3)^2\cos^2\phi} = 1$$

and that the two cylinders touch the sphere from within. Kiepert had in his Dissertation arrived at exactly similar results.

XI.

Although Lilienthal devoted a long chapter to a discussion the application of his method to Kiepert's curve $r^3 = \cos 3\phi$ (*vide.* 36) he does not mention that there is an analagous space curv Modifying somewhat his equations for the plane curve, let

$$
(1)\quad \left\{
\begin{aligned}
x &= A\frac{\sigma'}{\sigma}(u-a) + A'\frac{\sigma'}{\sigma}(u-a') + A_1\frac{\sigma'}{\sigma}(u-a_1) + A_1'\frac{\sigma'}{\sigma}(u-a_1') \\
&\qquad + A_2\frac{\sigma'}{\sigma}(u-a_2) + A_2'\frac{\sigma'}{\sigma}(u-a_2') \\
y &= B\frac{\sigma'}{\sigma}(u-a) + B'\frac{\sigma'}{\sigma}(u-a') + B_1\frac{\sigma'}{\sigma}(u-a_1) + B_1'\frac{\sigma'}{\sigma}(u-a_1') \\
&\qquad + B_2\frac{\sigma'}{\sigma}(u-a_2) + B_2'\frac{\sigma'}{\sigma}(u-a_2') \\
z &= C\frac{\sigma'}{\sigma}(u-a) + C'\frac{\sigma'}{\sigma}(u-a') + C_1\frac{\sigma'}{\sigma}(u-a_1) + C_1'\frac{\sigma'}{\sigma}(u-a_1') \\
&\qquad + C_2\frac{\sigma'}{\sigma}(u-a_2) + C_2'\frac{\sigma'}{\sigma}(u-a_2')
\end{aligned}
\right.
$$

where the accented quantities are the conjugates of the corresponding unaccented ones, and

$$
a = \frac{2\omega}{3},\quad a_1 = -\frac{2\omega}{3} + \frac{2\omega'}{3},\quad a_2 = -\frac{2\omega'}{3}
$$

$$
a' = \frac{2\omega'}{3},\quad a_1' = -\frac{2\omega'}{3} + \frac{2\omega}{3},\quad a_2' = -\frac{2\omega}{3}.
$$

Therefore, since $\frac{d}{du}\frac{\sigma'}{\sigma}(u) = \frac{d^2}{du^2}\log\sigma u = -\wp u$, we get

$$
\frac{dx}{du} = -A\wp(u-a) - A'\wp(u-a') - A_1\wp(u-a_1) - A_1'\wp(u-a_1')
$$
$$
- A_2\wp(u-a_2) - A_2'\wp(u-a_2').
$$

If we develop this in the neighbourhood of $u=a$ we get (see *Formeln* p. 10, and footnote p. 37).

$$
\begin{aligned}
\frac{dx}{du} = -\frac{A}{(u-a)^2} - \;.\;.\;. \; &- A'\wp(a-a') - (u-a)A'\wp'(a-a') - \;.\;.\;. \\
&- A_1\wp(a-a_1) - (u-a)A_1\wp'(a-a_1) - \;.\;.\;. \\
&- A_1'\wp(a-a_1') - (u-a)A_1'\wp'(a-a_1') - \;.\;.\;. \\
&- A_2\wp(a-a_2) - (u-a)A_2\wp'(a-a_2) - \;.\;.\;. \\
&- A_2'\wp(a-a_2') - (u-a)A_2'\wp'(a-a_2') - \;.\;.\;.
\end{aligned}
$$

$$\therefore \left(\frac{dx}{du}\right)^2 = \frac{A^2}{(u-a)^4} + \frac{2A}{(u-a)^2}\{A'p(a-a') + A_1p(a-a_1) + A_1'p(a-a_1') + A_2p(a-a_2) + A_2'p(a-a_2')\}$$
$$+ \frac{2A}{u-a}\{A'p'(a-a') + A_1p'(a-a_1) + A_1'p'(a-a_1') + A_2p'(a-a_2) + A_2'p'(a-a_2')\} + F(u-a)$$

where $F(u-a)$ is finite in the proximity of $u=a$.

Similarly in the neighbourhood of a_1 and a_2 we get

$$\left(\frac{dx}{du}\right)^2 = \frac{A_1^2}{(u-a_1)^4} + \frac{2A_1}{(u-a_1)^2}\{Ap(a_1-a) + A'p(a_1-a') + A_1'p(a_1-a_1') + A_2p(a_1-a_2) + A_2'p(a_1-a_2')\}$$
$$+ \frac{2A_1}{u-a_1}\{Ap'(a_1-a) + A'p'(a_1-a') + A_1'p'(a_1-a_1') + A_2p'(a_1-a_2) + A_2'p'(a_1-a_2')\} + F_1(u-a_1)$$

$$\left(\frac{dx}{du}\right)^2 = \frac{A_2^2}{(u-a_2)^4} + \frac{2A_2}{(u-a_2)^2}\{Ap(a_2-a) + A'p(a_2-a') + A_1p(a_2-a_1) + A_1'p(a_2-a_1') + A_2'p(a_2-a_2')\}$$
$$+ \frac{2A_2}{u-a_2}\{Ap'(a_2-a) + A'p'(a_2-a') + A_1p'(a_2-a_1) + A_1'p'(a_2-a_1') + A_2'p'(a_2-a_2')\} + F_2(u-a_2).$$

By putting B and C respectively for A we get the corresponding expressions for $\left(\frac{dy}{du}\right)^2$ and $\left(\frac{dz}{du}\right)^2$.

In order that the curve be algebraic the infinite terms in $\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$ must vanish, the conditions for which a

$$A^2 + B^2 + C^2 = 0 \qquad A_1^2 + B_1^2 + C_1^2 = 0 \qquad A_2^2 + B_2^2 + C_2^2 = 0$$

$$\left\{\begin{aligned}
&(AA' + BB' + CC')p(a-a') + (AA_1 + BB_1 + CC_1)p(a-a_1) + (AA_1' + BB_1' + CC_1')p(a-a_1') \\
&\qquad + (AA_2 + BB_2 + CC_2)p(a-a_2) + (AA_2' + BB_2' + CC_2')p(a-a_2') = 0 \\
&(A_1A + B_1B + C_1C)p(a_1-a) + (A_1A' + B_1B' + C_1C')p(a_1-a') + (A_1A_1' + B_1B_1' + C_1C_1')p(a_1-a_1') \\
&\qquad + (A_1A_2 + B_1B_2 + C_1C_2)p(a_1-a_2) + (A_1A_2' + B_1B_2' + C_1C_2')p(a_1-a_2') = 0 \\
&(A_2A + B_2B + C_2C)p(a_2-a) + (A_2A' + B_2B' + C_2C')p(a_2-a') + (A_2A_1 + B_2B_1 + C_2C_1)p(a_2-a_1) \\
&\qquad + (A_2A_1' + B_2B_1' + C_2C_1')p(a_2-a_1') + (A_2A_2' + B_2B_2' + C_2C_2')p(a_2-a_2') = 0
\end{aligned}\right.$$

and other three equations obtained by putting p′ instead of p in the last three.

The values of a, a_1, a_2, a', a_1', a_2' we have already stated.

Hence there are nine equations of condition, and these can be fulfilled since we have nine quantities at our disposal, viz.,

$$A\,B\,C \qquad A_1B_1C_1 \qquad A_2B_2C_2,$$

or their conjugates. So that corresponding to Kiepert's plane curve $r^3 = \cos 3\phi$ there exists a curve of double curvature whose arcs represent the elliptic integral of the first kind.

I shall prove the same thing by another method which will be found exceedingly useful in the next section.

$$\frac{dx}{du} = -A\mathrm{p}(u-a) - A'\mathrm{p}(u-a') - A_1\mathrm{p}(u-a_1) - A_1'\mathrm{p}(u-a_1')$$
$$- A_2\mathrm{p}(u-a_2) - A_2'\mathrm{p}(u-a_2')$$

For shortness write this as follows :

$$\frac{dx}{du} = -A\,\mathrm{p}(u-a) - P(u)$$

$$\text{or} \quad \frac{dx}{du} = -A_1\mathrm{p}(u-a_1) - P_1(u)$$

$$\text{or} \quad \frac{dx}{du} = -A_2\mathrm{p}(u-a_2) - P_2(u).$$

Similarly

$$\frac{dy}{du} = -B\mathrm{p}(u-a) - Q(u) = -B_1\mathrm{p}(u-a_1) - Q_1(u) = -B_2\mathrm{p}(u-a_2) - Q_2(u)$$

$$\frac{dz}{du} = -C\mathrm{p}(u-a) - R(u) = -C_1\mathrm{p}(u-a_1) - R_1(u) = -C_2\mathrm{p}(u-a_2) - R_2(u).$$

Then, in order to prove that the curve (1) is algebraic, we have to show that the conditions are satisfied that

$$\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$$

have no infinite terms for $u = a$, $u = a_1$, or $u = a_2$.

To show this we have to develop

$$\frac{dx}{du}, \quad \frac{dy}{du}, \quad \frac{dz}{du}$$

in the neighbourhood of the points a, a_1 and a_2.

For the development of $p(u-a)$ see Halphen's F.E., vol. I., p. 92, and for that of $P(u)$ see footnote, p. 37. Using these, we get for the proximity of $u=a$

$$(2)\quad \frac{dx}{du} = -\frac{A}{(u-a)^2} - Ac_0 - Ac_2(u-a)^2 - \ldots$$
$$\ldots - P(a) - (u-a)P'(a) - \frac{(u-a)^2}{2}P''(a) - \ldots$$

Similarly for the neighbourhoods of $u=a_1$ or $u=a_2$ we get

$$\frac{dx}{du} = -\frac{A_1}{(u-a_1)^2} - A_1c_0 - A_1c_2(u-a_1)^2 - \ldots$$
$$\ldots - P_1(a_1) - (u-a_1)P_1'(a_1) - \frac{(u-a_1)^2}{2}P''(a_1) - \ldots$$

$$\frac{dx}{du} = -\frac{A_2}{(u-a_2)^2} - A_2c_0 - A_2c_2(u-a_2)^2 - \ldots$$
$$\ldots - P_2(a_2) - (u-a_2)P_2'(a_2) - \frac{(u-a_2)^2}{2}P''(a_2) - \ldots$$

Squaring (2) we get

$$\left(\frac{dx}{du}\right)^2 = \frac{A^2}{(u-a)^4} + \frac{2A}{(u-a)^2}[Ac_0 + P(a)] + \frac{2A \,.\, P'(a)}{u-a} + F(u-a)$$

where $F(u-a)$ is finite in the neighbourhood of $u=a$.

By putting B and C respectively in place of A we get similar expressions for $\left(\frac{dy}{du}\right)^2$ and $\left(\frac{dz}{du}\right)^2$.

Then the necessary conditions that $\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$ be not infinite for $u=a$ are (where Σ denotes the sum of all similar terms got by interchanging A, B, and C, and P, Q, and R)

$$\Sigma A^2 = 0$$
$$\Sigma A[Ac_0 + P(a)] = 0$$
$$\Sigma A.P'(a) = 0$$

or we may write them

$$(3)\qquad \Sigma A^2 = 0, \qquad \Sigma A.P(a) = 0. \qquad \Sigma A.P'(a) = 0$$

Similarly in the neighbourhood of $u=a_1$ and $u=a_2$ respectively we get

(4) $\Sigma A_1^2=0, \quad \Sigma A_1 . P_1(a_1)=0, \quad \Sigma A_1 . P_1'(a_1)=0$

(5) $\Sigma A_2^2=0, \quad \Sigma A_2 . P_2(a_2)=0, \quad \Sigma A_2 . P_2'(a_2)=0.$

Thus we have in (3) (4) and (5) altogether nine equations of condition to be fulfilled in order that the given curve be algebraic, and these can be satisfied since we have nine quantities at our disposal, viz., A B C $A_1B_1C_1$ $A_2B_2C_2$ or their conjugates, and hence we come to the same conclusion as before.

XII.

Again, by Lilienthal's notation we can readily generalise Serret's plane curves and obtain a corresponding group of curves on a sphere.

We have seen (*vide* equation (16) p. 32 and footnote there) that for Serret's curves

$$\frac{dx+idy}{du}=\sum_{\nu=1}^{m} c_{1,\,\nu}\frac{d^{2\nu+1}}{du^{2\nu+1}}\log\sigma\left(u-\frac{\omega}{2}+\frac{\omega'}{2}\right)$$
$$+\sum_{\nu=1}^{n} c_{2,\,\nu}\frac{d^{2\nu+1}}{du^{2\nu+1}}\log\sigma\left(u+\frac{\omega}{2}-\frac{\omega'}{2}\right)$$

i.e.,

$$(1)\quad x+iy=\sum_{\nu=1}^{m} c_{1,\,\nu}\frac{d^{2\nu}}{du^{2\nu}}\log\sigma(u-a)+\sum_{\nu=1}^{n} c_{2,\,\nu}\frac{d^{2\nu}}{du^{2\nu}}\log\sigma(u-a_1)$$

where $a=\frac{\omega}{2}-\frac{\omega'}{2}$, and $a_1=-\frac{\omega}{2}+\frac{\omega'}{2}\cdot$

The curves having obviously two infinities, $u=a$ and $u=a_1$, therefore writing the expressions for x, y, and z in Lilienthal's notation [*cf.* X. (1)] we get

$$
\begin{aligned}
(2)\quad x = {} & A\,p(u-a) + A_2 p''(u-a) + \ldots\ldots + A_{2\nu}\, p^{(2\nu)}(u-a) \\
& + A'\,p(u-a') + A_2' p''(u-a') + \ldots\ldots + A'_{2\nu}\, p^{(2\nu)}(u-a') \\
& + A_1\,p(u-a_1) + A_3 p''(u-a_1) + \ldots\ldots + A_{2\nu+1} p^{(2\nu)}(u-a_1) \\
& + A_1' p(u-a_1') + A_3' p''(u-a_1') + \ldots\ldots + A'_{2\nu+1} p^{(2\nu)}(u-a_1')
\end{aligned}
$$

with similar expressions for y and z by putting B and C respectively in place of A.

For shortness, write the above

$$
(3)\left\{
\begin{aligned}
x &= A\,p(u-a) + A_2 p''(u-a) + \ldots\ldots + A_{2\nu}\, p^{(2\nu)}(u-a) + P(u) \\
& \quad \text{or} \\
x &= A_1 p(u-a_1) + A_3 p''(u-a_1) + \ldots\ldots + A_{2\nu+1} p^{(2\nu)}(u-a_1) + P_1(u) \\
& \quad \text{similarly} \\
y &= B\,p(u-a) + B_2 p''(u-a) + \ldots\ldots + B_{2\nu}\, p^{(2\nu)}(u-a) + Q(u) \\
&= B_1 p(u-a_1) + B_3 p''(u-a_1) + \ldots\ldots + B_{2\nu+1} p^{(2\nu)}(u-a_1) + Q_1(u) \\
z &= C\,p(u-a) + C_2 p''(u-a) + \ldots\ldots + C_{2\nu}\, p^{(2\nu)}(u-a) + R(u) \\
&= C_1 p(u-a_1) + C_3 p''(u-a_1) + \ldots\ldots + C_{2\nu+1} p^{(2\nu)}(u-a_1) + R_1(u)
\end{aligned}
\right.
$$

Comparing this abbreviated notation with the full expression for x given above, it is evident that $P\ Q\ R\ P_1\ Q_1\ R_1$ have only even derivatives.

Then, just as in the preceding sections, in order to prove that the curves for (3) are algebraic we have to show that the conditions are satisfied that

$$\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$$

be constant for $u = a$, or $u = a$.

Now,

$$
\begin{aligned}
\frac{dx}{du} = {} & Ap'(u-a) + A_2 p'''(u-a) + \ldots \\
& \ldots + A_{2\nu-2} p^{(2\nu-1)}(u-a) + A_{2\nu} p^{(2\nu+1)}(u-a) + P'(u).
\end{aligned}
$$

We have to develop this in the neighbourhood of $u = a$.

But

$$pu = \frac{1}{u^2} + c_0 + c_2u^2 + c_4u^4 + c_6u^6 + c_8u^8 + \cdots$$

(*vide* Halphen's F. E., vol. I. p. 92

$$\because\ p'u = -\frac{1.2}{u^3} + 2c_2u + 4c_4u^3 + 6c_6u^5 + 8c_8u^7 + \cdots$$

$$p''u = \frac{1.2.3}{u^4} + 2c_2 + 12c_4u^2 + 30c_6u^4 + 56c_8u^6 + \cdots$$

$$p'''u = -\frac{1.2.3.4}{u^5} + 24c_4u + 120c_6u^3 + 336c_8u^5 + \cdots$$

$$p^{\text{IV}}u = \frac{1.2.3.4.5}{u^6} + 24c_4 + 360c_6u^2 + 1680c_8u^4 + \cdots$$

$$p^{\text{V}}u = -\frac{1.2.3.4.5.6}{u^7} + 720c_6u + 6720c_8u^3 + \cdots$$

We may therefore include the even and odd derivatives of pu under the following useful general formula:—

$$p^{(2r)}\ u = \frac{(2r+1)\,!}{u^{2r+2}} + c_{0,\,2r} + c_{2,\,2r}u^2 + c_{4,\,2r}u^4 + \cdots$$

$$p^{(2r+1)}u = -\frac{(2r+2)\,!}{u^{2r+3}} + c_{1,\,2r+1}u + c_{3,\,2r+1}u^3 + c_{5,\,2r+1}u^5 + \cdots$$

Using the latter for the development of $\dfrac{dx}{du}$ above, we get

$$(4)\ \frac{dx}{du} = -\frac{2\text{A}}{(u-a)^3} - \frac{4\,!\text{A}_2}{(u-a)^5} - \cdots - \frac{(2\nu)\,!\text{A}_{2\nu-2}}{(u-a)^{2\nu+1}} - \frac{(2\nu+2)\,!\text{A}_{2\nu}}{(u-a)^{2\nu+3}}$$
$$+ (c_{1,\,1}\text{A} + c_{1,\,3}\text{A}_2 + \cdots + c_{1,\,2\nu+1}\text{A}_{2\nu})(u-a)$$
$$+ (c_{3,\,1}\text{A} + c_{3,\,3}\text{A}_2 + \cdots + c_{3,\,2\nu+1}\text{A}_{2\nu})(u-a)^3$$
$$\cdots\cdots\cdots$$
$$+ (c_{2\nu+1,\,1}\text{A} + c_{2\nu+1,\,3}\text{A}_2 + \cdots + c_{2\nu+1,\,2\nu+1}\text{A}_{2\nu})(u-a)^{2\nu+1}$$
$$+ (u-a)\text{P}''(a) + \frac{(u-a)^3}{3\,!}\text{P}^{\text{IV}}(a) + \cdots$$
$$\cdots + \frac{(u-a)^{2\nu+1}}{(2\nu+1)\,!}\text{P}^{(2\nu+2)}(a).*$$

* For the development of $\text{P}'(u)$ in the proximity of $u=a$, *cf.* footnote, p. 37, keeping in mind that $\text{P}(u)$ has only even derivatives as already pointed out.

Similarly, for the neighbourhood of $u=a_1$, we get

$$(5)\quad \frac{dx}{du}=-\frac{2A_1}{(u-a_1)^3}-\frac{4!A_3}{(u-a_1)^5}-\ \dots\ -\frac{(2\nu)!A_{2\nu-1}}{(u-a_1)^{2\nu+1}}-\frac{(2\nu+2)!A_{2\nu+1}}{(u-a_1)^{2\nu+3}}$$
$$+(\ c_{1,1}A_1+\ c_{1,3}A_3+\ \dots\ +\ c_{1,2\nu+1}A_{2\nu+1})(u-a_1)$$
$$+(\ c_{3,1}A_1+\ c_{3,3}A_3+\ \dots\ +\ c_{3,2\nu+1}A_{2\nu+1})(u-a_1)^3$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$+(c_{2\nu+1,1}A_1+c_{2\nu+1,3}A_3+\ \dots\ +c_{2\nu+1,2\nu+1}A_{2\nu+1})(u-a_1)^{2\nu+1}$$
$$+(u-a_1)P_1''(a_1)+\frac{(u-a_1)^3}{3!}P_1^{IV}(a_1)+\ \dots$$
$$\dots\ +\frac{(u-a_1)^{2\nu+1}}{(2\nu+1)!}P_1^{(2\nu+2)}(a_1).$$

On replacing A by B and C respectively, we get the corresponding expressions for $\dfrac{dy}{du}$ and $\dfrac{dz}{du}$.

Now, in order that the condition

$$\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2+\left(\frac{dz}{du}\right)^2=\mathit{constant}$$

be fulfilled in the neighbourhood of $u=a$, and $u=a_1$, it is of course necessary that the terms containing $u-a$ and $u-a_1$ in the denominator vanish. On squaring (4) and (5) we see that the necessary conditions for this are (where Σ denotes the sum of all similar terms got by interchanging A, B, and C, and P, Q, and R)

$$(6)\quad \begin{cases} \Sigma A^2=0, \qquad \Sigma A_2^2=0,\ \dots\ \Sigma A_{2\nu}^2=0 \\ \Sigma A\,(A_2+A_4+\ \dots\ +A_{2\nu})=0 \\ \Sigma A_2(A_4+A_6+\ \dots\ +A_{2\nu})=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \Sigma A_{2\nu-2}A_{2\nu}=0 \\ \Sigma P''(a)[2\ A+4!A_2+\ \dots\ +(2\nu)!A_{2\nu-2}+(2\nu+2)!A_{2\nu}]=0 \\ \Sigma P^{IV}(a)[4!A_2+6!A_4+\ \dots\ +(2\nu)!A_{2\nu-2}+(2\nu+2)!A_{2\nu}]=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \Sigma P^{(2\nu+2)}(a)A_{2\nu}=0, \end{cases}$$

and similar equations to be satisfied for the neighbourhood of $u=a_1$. But from (2) we see that

(7) $\frac{A_2}{A}=\frac{B_2}{B}=\frac{C_2}{C}=\lambda$, say ; and $\frac{A_4}{A_2}=\frac{B_4}{B_2}=\frac{C_4}{C_2}$, and so on.

Hence we may put

$$(8)\quad \begin{cases} A_2=\lambda A, & A_4=\lambda_1 A \;\ldots\; A_{2\nu}=\lambda_{\nu-1}A. \\ B_2=\lambda B, & B_4=\lambda_1 B \;\ldots\; B_{2\nu}=\lambda_{\nu-1}B. \\ C_2=\lambda C, & C_4=\lambda_1 C \;\ldots\; C_{2\nu}=\lambda_{\nu-1}C. \end{cases}$$

Therefore (6) reduces to

(9) $\Sigma A^2=0,\quad \Sigma A.P''(a)=0,\quad \Sigma A.P^{IV}(a)=0,\ldots \Sigma A.P^{(2\nu+2)}(a)=0.$

Similarly, for the neighbourhood of $u=a_1$, on putting

$$(10)\quad \begin{cases} A_3=\mu A_1, & A_5=\mu_1 A_1 \;\ldots\; A_{2\nu+1}=\mu_{\nu-1}A_1 \\ B_3=\mu B_1, & B_5=\mu_1 B_1 \;\ldots\; B_{2\nu+1}=\mu_{\nu-1}B_1 \\ C_3=\mu C_1, & C_5=\mu_1 C_1 \;\ldots\; C_{2\nu+1}=\mu_{\nu-1}C_1 \end{cases}$$

we get

(11) $\Sigma A_1^2=0,\ \Sigma A_1.P_1''(a_1)=0,\ \Sigma A_1.P_1^{IV}(a_1)=0\ldots \Sigma A_1.P_1^{(2\nu+2)}(a_1)=0.$

We thus have in (9) and (11) altogether $2\nu+4$ equations among $2\nu+6$ quantities

$$A\,B\,C\quad A_1B_1C_1\quad \lambda\,\lambda_1\ldots\lambda_{\nu-1}\,\mu\,\mu_1\ldots\mu_{\nu-1}.$$

But if we further stipulate that the curves be spherical ones, this is equivalent to other two conditions, for, proceeding in the same way as Lilienthal does for the spherical lemniscate, we have to find the conditions that the expression $(x-a)^2+(y-\beta)^2+(z-\gamma)^2$ be constant, *i.e.*, have no infinities for $u=a$, $u=a_1$.

Now, developing in the proximity of $u=a$ the value of x given in (3) we get (*cf.* footnote, p. 37, and equation (4) above)

$$x=+\frac{A}{(u-a)^2}+\frac{3!A_2}{(u-a)^4}+\ldots\ldots+\frac{(2\nu+1)!A_{2\nu}}{(u-a)^{2\nu+2}}\ldots$$

$$+\;P(a)\;+\frac{(u-a)^2P''(a)}{2}+\ldots\ldots+\frac{(u-a)^{2\nu}P^{(2\nu)}(a)}{(2\nu)!}.$$

Wherefore, keeping (8) in mind, we have

$$x - a = + \frac{A}{(u-a)^2} + \frac{3!\lambda A}{(u-a)^4} + \ldots\ldots + \frac{(2\nu+1)!\lambda_{\nu-1}A}{(u-a)^{2\nu+2}} \ldots\ldots$$

$$+ P(a) - a + \frac{(u-a)^2 P''(a)}{2} + \ldots\ldots + \frac{(u-a)^{2\nu}P^{(2\nu)}(a)}{(2\nu)!}.$$

Similarly, in the neighbourhood of $u = a_1$

$$x - a = + \frac{A_1}{(u-a_1)^2} + \frac{3!\mu A_1}{(u-a_1)^4} + \ldots + \frac{(2\nu+1)!\mu_{\nu-1}A_1}{(u-a_1)^{2\nu+2}} \ldots$$

$$+ P_1(a_1) - a + \frac{(u-a_1)^2 P_1''(a_1)}{2} + \ldots + \frac{(u-a_1)^{2\nu}P^{(2\nu)}(a_1)}{(2\nu)!}.$$

We get the corresponding expressions for $y - \beta$ and $z - \gamma$ on replacing A by B and C respectively.

Then, on squaring, we find that in order that the curve be spherical, *i.e.*, in order that $(x-a)^2 + (y-\beta)^2 + (z-\gamma)^2$ have no infinities at $u = a$ and $u = a_1$, the terms containing $u - a$ and $u - a_1$ in the denominator must vanish.

Therefore, in addition to equations of the types in (9) and (11) we also get

$$(12) \quad \left\{ \begin{array}{l} A\,[P\,(a\,) - a] + B\,[Q\,(a\,) - \beta] + C\,[R\,(a\,) - \gamma] = 0 \\ A_1[P_1(a_1) - a] + B_1[Q_1(a_1) - \beta] + C_1[R_1(a_1) - \gamma] = 0. \end{array} \right.$$

Hence we have altogether $2\nu + 6$ equations of condition and and $2\nu + 6$ quantities at our disposal, so that the necessary conditions can be satisfied that, corresponding to Serret's infinite group of plane curves we have an infinite group of spherical curves whose arcs are elliptic integrals of the first kind.

Finally, I wish to prove that corresponding to Kiepert's infinity of plane curves there is an infinity of spherical ones whose arcs represent the first elliptic integral.

The equation for Kiepert's curves is (see p. 29)

$$\frac{dx + idy}{du} = \sum_{\nu=1}^{r_1} c_{1,\nu} \frac{d^{\nu+1}\log\sigma(u-a_1)}{du^{\nu+1}} + \ldots + \sum_{\nu=1}^{r_m} c_{m,\nu} \frac{d^{\nu+1}\log\sigma(u-a_m)}{du^{\nu+1}}$$

i.e.,

$$(1^a)\quad x+iy=\sum_{\nu=1}^{r_1} c_{1,\nu}\frac{d^\nu \log\sigma(u-a_1)}{du^\nu}+\ldots+\sum_{\nu=1}^{r_m} c_{m,\nu}\frac{d^\nu \log\sigma(u-a_m)}{du^\nu}$$

where the infinities $u=a_1, a_2, \ldots\ldots a_m$ have any given values.

We shall call the equations used (1^a), (2^a), etc., to indicate that they are the analagous ones to (1), (2), etc., in the previous part of the section.

Since the points at infinity are $u=a_1, a_2, \ldots\ldots a_m$ we may represent the coordinates of Kiepert's curves thus:—

$$\begin{aligned}(2^a)\quad x= &\ A_{1,0}\mathrm{p}(u-a_1)+A_{1,1}\mathrm{p}'(u-a_1)+\ \ldots\ +A_{1,\nu}\mathrm{p}^{(\nu)}(u-a_1)\\ &+A'_{1,0}\mathrm{p}(u-a_1')+A'_{1,1}\mathrm{p}'(u-a'_1)+\ \ldots\ +A'_{1,\nu}\mathrm{p}^{(\nu)}(u-a'_1)\\ &+A_{2,0}\mathrm{p}(u-a_2)+A_{2,1}\mathrm{p}'(u-a_2)+\ \ldots\ +A_{2,\nu}\mathrm{p}^{(\nu)}(u-a_2)\\ &+A'_{2,0}\mathrm{p}(u-a_2')+A'_{2,1}\mathrm{p}'(u-a'_2)+\ \ldots\ +A'_{2,\nu}\mathrm{p}^{(\nu)}(u-a'_2)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &+A_{m,0}\mathrm{p}(u-a_m)+A_{m,1}\mathrm{p}'(u-a_m)+\ \ldots\ +A_{m,\nu}\mathrm{p}^{(\nu)}(u-a_m)\\ &+A'_{m,0}\mathrm{p}(u-a'_m)+A'_{m,1}\mathrm{p}'(u-a'_m)+\ \ldots\ +A'_{m,\nu}\mathrm{p}^{(\nu)}(u-a'_m)\end{aligned}$$

and similar expressions for y and z by putting B and C respectively for A.

The contracted notation corresponding to (3) will be

(3^a)

$$\left\{\begin{aligned} x&=A_{1,0}\mathrm{p}(u-a_1)+A_{1,1}\mathrm{p}'(u-a_1)+\ \ldots\ +A_{1,\nu}\mathrm{p}^{(\nu)}(u-a_1)+P_1(u)\\ &=A_{2,0}\mathrm{p}(u-a_2)+A_{2,1}\mathrm{p}'(u-a_2)+\ \ldots\ +A_{2,\nu}\mathrm{p}^{(\nu)}(u-a_2)+P_2(u)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &=A_{m,0}\mathrm{p}(u-a_m)+A_{m,1}\mathrm{p}'(u-a_m)+\ \ldots\ +A_{m,\nu}\mathrm{p}^{(\nu)}(u-a_m)+P_m(u)\\ y&=B_{1,0}\mathrm{p}(u-a_1)+B_{1,1}\mathrm{p}'(u-a_1)+\ \ldots\ +B_{1,\nu}\mathrm{p}^{(\nu)}(u-a_1)+Q_1(u)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &=B_{m,0}\mathrm{p}(u-a_m)+B_{m,1}\mathrm{p}'(u-a_1)+\ \ldots\ +B_{m,\nu}\mathrm{p}^{(\nu)}(u-a_m)+Q_m(u)\\ z&=C_{1,0}\mathrm{p}(u-a_1)+C_{1,1}\mathrm{p}'(u-a_1)+\ \ldots\ +C_{1,\nu}\mathrm{p}^{(\nu)}(u-a_1)+R_1(u)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &=C_{m,0}\mathrm{p}(u-a_m)+C_{m,1}\mathrm{p}'(u-a_1)+\ \ldots\ +C_{m,\nu}\mathrm{p}^{(\nu)}(u-a_m)+R_m(u).\end{aligned}\right.$$

Therefore

$$\frac{dx}{du} = A_{1,0}\wp'(u-a_1) + A_{1,1}\wp''(u-a_1) + \dots$$
$$\dots + A_{1,\nu-1}\wp^{(\nu)}(u-a_1) + A_{1,\nu}\wp^{(\nu+1)}(u-a_1) + P_1'(u).$$

Developing this in the neighbourhood of $u = a_1$ we get, since the derivatives are alternately odd and even (*vide* the general expressions for the odd and even derivatives of $\wp u$ on p. 56),

$$\begin{aligned}
(4a)\quad \frac{dx}{du} = &-\frac{2A_{1,0}}{(u-a_1)^3} + \frac{3!A_{1,1}}{(u-a_1)^4} - \dots + \frac{(\nu+1)!A_{1,\nu-1}}{(u-a_1)^{\nu+2}} - \frac{(\nu+2)!A_{1,\nu}}{(u-a_1)^{\nu+3}}{}^{*}\\
&+ (c_{0,2}A_{1,1} + c_{0,4}A_{1,3} \quad + \dots + c_{0,\nu}A_{1,\nu-1})\\
&+ (c_{2,2}A_{1,1} + c_{2,4}A_{1,3} \quad + \dots + c_{2,\nu}A_{1,\nu-1})(u-a_1)^2\\
&\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad,\quad\cdot\quad\cdot\\
&+ (c_{\nu,2}A_{1,1} + c_{\nu,4}A_{1,3} \quad + \dots + c_{\nu,\nu}A_{1,\nu-1})(u-a_1)^\nu\\
&+ (c_{1,1}A_{1,0} + c_{1,3}A_{1,2} \quad + \dots + c_{1,\nu+1}A_{1,\nu})(u-a_1)\\
&+ (c_{3,1}A_{1,0} + c_{3,3}A_{1,2} \quad + \dots + c_{3,\nu+1}A_{1,\nu})(u-a_1)^3\\
&\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&+ (c_{\nu+1,1}A_{1,0} + c_{\nu+1,3}A_{1,2} + \dots + c_{\nu+1,\nu+1}A_{1,\nu})(u-a_1)^{\nu+1}\\
&+ P_1'(a_1) + (u-a_1)P_1''(a_1) + \frac{(u-a_1)^2}{2}P_1'''(a_1) \dots\\
&\qquad\qquad \dots + \frac{(u-a_1)^\nu}{\nu!}P_1^{(\nu+1)}(a_1)
\end{aligned}$$

and corresponding expressions for $\frac{dx}{du}$ in the neighbourhoods of

$$u = a_2,\ u = a_3,\ \dots\ u = a_m.$$

Also, putting B and C respectively in place of A we get similar expansions for $\frac{dy}{du}$ and $\frac{dz}{du}$.

Then expressing the conditions that $\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$

---

* Taking for granted that ν is even; of course it does not matter whether it is odd or even—it is merely a question of which sign we shall use.

may have no infinities at $u=a_1$ we get

$$
(6a)\quad \begin{cases}
\Sigma A^2_{1,0}=0,\ \ \Sigma A^2_{1,1}=0,\ \ \Sigma A^2_{1,2}=0,\ \ldots\ \Sigma A^2_{1,\nu}=0 \\
\Sigma A_{1,0}(A_{1,1}+A_{1,2}\ +\ \ldots\ +A_{1,\nu})=0 \\
\Sigma A_{1,1}(A_{1,2}+A_{1,3}\ +\ \ldots\ +A_{1,\nu})=0 \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\Sigma A_{1,\nu-1}A_{1,\nu}=0 \\
\Sigma P_1'(a_1)[-2A_{1,0}\ +3!A_{1,1}-4!A_{1,2}+\ \ldots\ -(\nu+2)!A_{1,\nu}]=0 \\
\Sigma P_1''(a_1)[-2A_{1,0}\ +3!A_{1,1}-4!A_{1,2}+\ \ldots\ -(\nu+2)!A_{1,\nu}]=0 \\
\Sigma P_1'''(a_1)[-2A_{1,0}\ +3!A_{1,1}-4!A_{1,2}+\ \ldots\ -(\nu+2)!A_{1,\nu}]=0 \\
\Sigma P_1^{\mathrm{IV}}(a_1)[+3!A_{1,1}-4!A_{1,2}\ \ +\ \ldots\ -(\nu+2)!A_{1,\nu}]=0 \\
\Sigma P_1^{\mathrm{V}}(a_1)[-4!A_{1,2}+5!A_{1,3}\ \ -\ \ldots\ -(\nu+2)!A_{1,\nu}]=0 \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\Sigma P_1^{(\nu+1)}(a_1).A_{1,\nu}=0.
\end{cases}
$$

But

$$
(7a)\qquad \frac{A_{1,1}}{A_{1,0}}=\frac{B_{1,1}}{B_{1,0}}=\frac{C_{1,1}}{C_{1,0}}=\lambda_{1,0},\ \text{say; and}\ \ \frac{A_{1,2}}{A_{1,1}}=\frac{B_{1,2}}{B_{1,1}}=\frac{C_{1,2}}{C_{1,1}},
$$

and so on.

Hence we may put

$$
(8a)\quad \begin{cases}
A_{1,1}=\lambda_{1,1}A_{1,0}, & A_{1,2}=\lambda_{1,2}A_{1,0},\ \ldots\ldots\ A_{1,\nu}=\lambda_{1,\nu}A_{1,0} \\
B_{1,1}=\lambda_{1,1}B_{1,0}, & B_{1,2}=\lambda_{1,2}B_{1,0},\ \ldots\ldots\ B_{1,\nu}=\lambda_{1,\nu}B_{1,0} \\
C_{1,1}=\lambda_{1,1}C_{1,0}, & C_{1,2}=\lambda_{1,2}C_{1,0},\ \ldots\ldots\ C_{1,\nu}=\lambda_{1,\nu}C_{1,0}.
\end{cases}
$$

Therefore (6a) reduces to

(9a)

$$
\Sigma A^2_{1,0}=0,\ \ \Sigma A_{1,0}P_1'(a_1)=0,\ \ \Sigma A_{1,0}P_1''(a_1)=0\ \ldots\ \Sigma A_{1,0}P_1^{(\nu+1)}(a_1)=0.
$$

Similarly for the points $u=a_2,\ u=a_3,\ \ldots\ u=a_m$, we get

$$
\begin{cases}
\Sigma A^2_{2,0}=0, & \Sigma A_{2,0}P_2'(a_2)=0,\ \ldots\ldots\ \Sigma A_{2,0}P_2^{(\nu+1)}(a_2)=0 \\
\Sigma A^2_{3,0}=0, & \Sigma A_{3,0}P_3'(a_3)=0,\ \ldots\ldots\ \Sigma A_{3,0}P_3^{(\nu+1)}(a_3)=0 \\
\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\Sigma A^2_{m,0}=0, & \Sigma A_{m,0}P'_m(a_m)=0,\ \ldots\ldots\ \Sigma A_{m,0}P_m^{(\nu+1)}(a_m)=0
\end{cases}
$$

In order that the curves be algebraic we thus have to satisfy the above $(\nu+2)m$ equations of condition among the $(\nu+3)m$ quantities :—

$$\begin{array}{cccccc} A_{1,0} & B_{1,0} & C_{1,0} & \lambda_{1,1} & \lambda_{1,2} \cdots\cdots & \lambda_{1,\nu} \\ A_{2,0} & B_{2,0} & C_{2,0} & \lambda_{2,1} & \lambda_{2,2} \cdots\cdots & \lambda_{2,\nu} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{m,0} & B_{m,0} & C_{m,0} & \lambda_{m,1} & \lambda_{m,2} \cdots\cdots & \lambda_{m,\nu} . \end{array}$$

But if the curves be spherical ones this will introduce other m conditions corresponding to (12) viz.,

$$(12a)\left\{\begin{array}{l} A_{1,0}[P_1(a_1)-\alpha]+B_{1,0}[Q_1(a_1)-\beta]+C_{1,0}[R_1(a_1)-\gamma]=0 \\ A_{2,0}[P_2(a_2)-\alpha]+B_{2,0}[Q_2(a_2)-\beta]+C_{2,0}[R_2(a_2)-\gamma]=0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ A_{m,0}[P_m(a_m)-\alpha]+B_{m,0}[Q_m(a_m)-\beta]+C_{m,0}[R_m(a_m)-\gamma]=0. \end{array}\right.$$

We have thus altogether $(\nu+3)m$ equations of condition and these can be satisfied as we have $(\nu+3)m$ quantities at our disposal, so that we have an infinity of algebraic spherical curves similar to Kiepert's plane curves and whose arcs likewise represent the elliptic integral of the first kind.

Kiepert asserted that to all the curves of single curvature whose arcs are elliptic integrals of the first kind there exist analogous curves of double curvature whose arcs are also first elliptic transcendentals. He says (*De curvis quarum arcus*, etc., p. 23), "omnes illæ curvæ, quæ in plano possunt investigari, singulares tantum casus sunt curvarum duplicis curvaturæ." He only proved his assertion, however, in the particular case of the lemniscate which Roberts had done about 30 years before, and even to that case he could not apply the method he had discovered for plane curves. We now see that by generalising Lilienthal's method for

plane curves, the feature of which is that it takes into account the behaviour of a curve at its singular points, we can prove the complete truth of Kiepert's statement. I have shown this in the case of the curve $r^3 = \cos 3\phi$ and, indeed, of Kiepert's plane curves in general which include Serret's infinity of curves and all the others as particular cases. These results, so far as I know, have not been obtained before, and they seem to be of considerable importance as they supply what was awanting in order to complete the geometrical representation of elliptic integrals of the first kind.

On the Foundations of Dynamics.

By Dr PEDDIE.

Note on a Theorem in connection with the Hessian of a Binary Quantic.

By CHARLES TWEEDIE, M.A., B.Sc.

Extension of the "Medial Section" problem (Euclid II:11, VI:30, etc.) and derivation of a Hyperbolic Graph.

By R. E. ANDERSON, M.A.

To divide the straight line AB (containing a units) at C so that

$$AB \, . \, BC = p \, . \, AC^2.$$

§ I.

By algebra, taking the positive root,

$$AC = \frac{AB}{2p}(\sqrt{4p+1} - 1), \qquad . \quad . \quad . \quad (1.)$$

The number p may therefore have any positive value, integral or fractional, and when negative cannot exceed $\frac{1}{4}$. Secondly, AC and AB are incommensurable except when $4p+1$ is a square:— *e.g.*, if $4p = (q-1)(q+1)$ or if $p = q(q+1)$, q being any positive integer or fraction.

To find the surd-line $\sqrt{4p+1}$ geometrically is the heart of the problem. *Euclid* solves it (II:11) when $p=1$ by I : 47, which is also used in Ex. i., ii., iii. following; but II:14 will sometimes be easier. Since equation (1.) becomes $AC = \sqrt{4p+1} - 1$, if $AB = 2p$, *i.e.*, if the unit line is $\dfrac{AB}{2p}$ or $\dfrac{AM}{p}$*, we construct thus:—

(Figure 1.)

Ex. i. Solve $AB.BC=3AC^2$.* Here $p=3$, surd $=\sqrt{13}$.
Take $AK=\frac{1}{3}AM$, $AR=2AK$, and $KS=RM$.
C the point required is determined by a square described on AS.

(Figure 2.)

Ex. ii. Solve $AB.BC=4AC^2$. Here $p=4$, surd $=\sqrt{17}$.
Take $AK=\frac{1}{4}AM$ and $KS=KM$.
C the point required is determined by square on AS.

(Figure 3.)

Ex. iii. Solve $AB.BC=7AC^2$. Here $p=7$, surd $=\sqrt{29}$.
Take $AK=\frac{1}{7}AM$, $AR=2AK$, $AT=5AK$, and $KS=RT$.
C is determined by square on AS.

(Figure 4.)

Ex. iv. Solve $AB.BC=AC^2$ by II : 14. Here $p=1$, surd $=\sqrt{5}$. Produce AB both ways till $AK=AM$ the unit, and $AH=5AM$ $\therefore AR=\sqrt{5}$ if $\perp AB$ and limited by $\frac{1}{2}\odot$ on KH : $\therefore$ if $KC=AR$, C gives the medial section of AB; and if $KC'=KC$, C′ is point of external section, corresponding to the negative root of equation (1.).

(Figure 5.)

Ex. v. Solve $AB.BC=\frac{2}{3}AC^2$. Here $p=\frac{2}{3}$, surd $=\sqrt{\frac{11}{3}}=\frac{1}{3}\sqrt{33}$. Produce AB both ways. In AB produced take $AK=\frac{3}{2}AM$, $AW=3AK$, and $AY=11AK$. $\therefore AR=\sqrt{3\times 11}$ if $\perp AB$ and limited by $\frac{1}{2}\odot$ on WY. Hence if $KC=\frac{1}{3}AR$, C is the point required; and if $KC'=KC$, C′ is the point of external section, as in Ex. iv. above.

Thus for a given value of p the surd number $\sqrt{4p+1}$ though intractable to Arithmetic can always be found by Geometry. In certain cases lower surds should be subsidised: thus for $\sqrt{33}$ (just found by II : 14) we may say $33=5^2+(2\sqrt{2})^2$.
Similarly $21=4^2+(\sqrt{5})^2$; $181=13^2+(2\sqrt{3})^2$.

---

* In the diagrams M is the mid-point of AB and in Figs. 1, 2, 3, SAK is $\perp$AB.

§ II.

(Figure 6.)

For the general case of equation (1.), the surd line $\sqrt{4p+1}$ can be expressed either by I : 47, since $4p+1=(2\sqrt{p})^2+1^2$, or by II : 14. Thus, by the latter, produce AB both ways till $AK=$ the unit $=\dfrac{AB}{2p}$ and $BH=BK$, so that $AH=4p+1$ units. $\therefore$ $AR=\sqrt{4p+1}$ if $\perp$ AB and limited by $\frac{1}{2}\odot$ on KH. Hence if $KC=KC'=AR$, C and C′ are the required points of section.

(Figure 7.)

Mr G. Duthie suggests a third general construction for eq. (1.). "Produce AB till $BK=MB=MA$, and take $MH=\dfrac{AM}{p}$: with centre K and radius KM describe $\odot$ MQW. Finally take $HR=HQ\cdot$ the tangent from H. Then if $AC=MR$, C is the point required."

As shown above, C and C′ are two points in AB or its production which determine the roots of the original quadratic. Thus, if $AR=\dfrac{AB}{2p}$, then

(α.), when p is positive, with limits ∞ and 0,

$$AC=AR(\sqrt{1+4p}-1), \text{ and } AC'=-AR(\sqrt{1+4p}+1),$$

$\therefore$ as AR grows continuously from 0 to ∞, so C and C′ move further apart from A in opposite directions;

(β.), when p is negative, with limits $-\frac{1}{4}$ and $-\frac{1}{\infty}$,

$$AC=AR(1-\sqrt{1+4p}), \text{ and } AC'=AR(1+\sqrt{1+4p}),$$

$\therefore$ as AR grows continuously from $-2a$ to $-\infty$, so C and C′ move in opposite directions further and further apart from Z which is a point in AB produced positively so that $AZ=2a$.

(γ.). At any instant both for (α.) and (β.), the distance $CC'=$ twice the surd-line; and, numerically, $AC'-AC=2AR$.

We may note also from equation (1.) that generally $\frac{AC}{AM} = \frac{\sqrt{4p+1}-1}{p}$; $\therefore$ $AC \lesseqgtr AM$ according as $\sqrt{4p+1} \lesseqgtr 1+p$, or as $2 \lesseqgtr p$: [Thus $AC = \frac{1}{2}AB$ when $p=2$; $AC > \frac{1}{2}AB$ when $p<2$ as in *Euclid* II: 11 and Ex. v. above; $AC < \frac{1}{2}AB$ when $p>2$ as Ex. i., ii., iii.], ... (2.)

§ III.

(Figure 8.)

To show graphically the variation of the segment AC, as obtained by equation (1.), I have placed P_1R_1 P_2R_2 P_3R_3 . . . . perpendicular to the fixed line AB, so that

$P_1R_1 = AR_1(\sqrt{5}-1)$, $P_2R_2 = AR_2(\sqrt{9}-1)$, $P_3R_3 =$ etc., etc.,

and generally $PR = AR(\sqrt{4p+1}-1)$, where $PR = AC$, $AR = \frac{AB}{2p}$

$$\therefore \qquad PR + AR = PS = AR\sqrt{4p+1} = \frac{AS}{\sqrt{2}}\sqrt{4p+1}$$

$$\therefore \qquad \left.\begin{aligned} 2PS^2 &= AS^2(4p+1) \\ &= AS\,(2a\sqrt{2}+AS) \end{aligned}\right\} \text{ since } 4p = \frac{2a}{AR} = \frac{2a\sqrt{2}}{AS},$$

$$= AS.A'S, \qquad . \; . \; . \; . \; . \; . \; . \; . \qquad (3.)$$

Hence for P, any point in the locus, the square of PS has a constant ratio to the rectangle AS . A′S, and that is the geometrical property of a Hyperbola having A′OASE as a diameter and PS an ordinate to it. Thus $PS = SQ$, FAD is a tangent at A, F′A′ another at A′, and O is the centre of the curve.

With reference to the original problem, PS (or SQ) is the surd-line $\sqrt{4p+1}$, AR (or RS) the unit-line, and the two roots of equation (1.) are PR and RQ. If, for example,

$$\left.\begin{aligned} &p = 1,\ SP = SQ = \sqrt{5} \\ &\quad PR = \ \ \sqrt{5}-1 = AC, \text{ internal segment} \\ &\quad RQ = -\sqrt{5}-1 = AC', \text{ external segment} \end{aligned}\right\} \begin{aligned} &\textit{cf. } \text{Ex. iv. of § I.} \\ &\text{and } \textit{Euclid} \text{ II : 11.} \end{aligned}$$

The ordinate at P′ if produced upward to meet the branch P′V′A′ in Q′, and downward to meet AB in R′, gives $P'R' = AC$ and $Q'R' = AC'$, for p negative. The Euclidian solution therefore of this case must place C and C′ in AB produced through B, as already shown, § II.

§ IV.

The Cartesian equation to this Hyperbola is at once derived from equation (3.) thus, choosing OFG as the axis of x,

$$2PS^2 = AS \,.\, A'S = (OS - a\sqrt{2})(OS + a\sqrt{2})$$

$$\left.\begin{aligned} &= OS^2 - 2a^2 \\ \therefore\quad 2(x+y)^2 &= 2x^2 - 2a^2 \end{aligned}\right\} \text{ putting } x+y \text{ for PS and } x\sqrt{2} \text{ for OS.}$$

$$\therefore\qquad y^2 + 2xy + a^2 = 0. \quad . \quad . \quad . \quad . \quad (4.)$$

The form of this equation shows that one asymptote is parallel to the axis of x $\therefore$ OFG is that asymptote. But $y^2 + 2xy = y(y + 2x)$, $\therefore$ $y + 2x = 0$ is the other asymptote, viz., the line OD. Thus V′OV bisecting the angle FOD is the transverse axis of the Hyperbola, V and V′ are the vertices.

Finally, referring the curve to its own axes, equation (4.) becomes

$$\frac{x^2}{\sqrt{5}+1} - \frac{y^2}{\sqrt{5}-1} = \frac{a^2}{2}, \quad \text{or} \quad \left(\frac{x}{h}\right)^2 - \left(\frac{y}{k}\right)^2 = 1, \quad . \quad (5.)$$

where

$$h^2 = OV^2 = \frac{a^2}{2}(\sqrt{5}+1) = 2a^2\cos\frac{\pi}{5}$$

$$k^2 = OW^2 = \frac{a^2}{2}(\sqrt{5}-1) = 2a^2\cos\frac{2\pi}{5}$$

Thus $hk = a^2 = \frac{1}{4}LL'$, where L is the lat. rectum of the curve, and L′ that of its conjugate. Hence

$$L = 4k\cos 72°.$$

A curious property therefore of our Hyperbola is that

$$\frac{L}{WW'} = \frac{WW'}{VV'} = \frac{VV'}{L'} = 2\cos 72°, \quad . \quad . \quad (6.)$$

In other words, an isosceles Δ satisfying *Euclid* IV : 10 is found by taking the two terms of any one of those three fractions, and making the numerator the base.

Another singular property, easily deduced, is that

$$r^{\frac{2}{3}} + a^{\frac{2}{3}} = (4L)^{\frac{2}{3}}, \quad . \quad . \quad . \quad . \quad (7)$$

where r = radius of curvature at the extremity of L the latus rectum.

Second Meeting, 11th December 1896.

This meeting was postponed as the funeral of the President, Rev. John Wilson, M.A., F.R.S.E., took place that day.

Third Meeting, January 8th, 1897.

J. B. CLARK, Esq., M.A., F.R.S.E., Vice-President, in the Chair.

Theorems on Normals of an Ellipse.

By Professor A. H. ANGLIN.

The condition that the normals at the points whose eccentric angles are α, β, γ *shall be concurrent is*

$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0.$$

The following method of establishing this result, as compared with those given in works on *Concis*, is direct, and also has the advantage of simplicity, by first proving the Trigonometrical identity

$$\sin 2\alpha\sin(\beta-\gamma)+\ \ldots\ =4\sin\frac{\beta-\gamma}{2}\ \ldots\ \{\sin(\beta+\gamma)+\ \ldots\ \},\quad (\mathrm{A})$$

which may be simply done by multiplying both sides of the well-known identity

$$\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta)$$
$$=-4\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}$$

by $$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta),$$

when it will be easily found that the product in the first member reduces to that in (A) with opposite sign.

Now, writing the equation to the normal at ϕ in the form

$$ax\sin\phi-by\cos\phi=\frac{c^2}{2}\sin 2\phi$$

the normals at α, β, γ are concurrent if (eliminating x, y as usual)

$$\begin{vmatrix} \sin 2\alpha, & \sin\alpha, & \cos\alpha \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = 0$$

that is, if $$\sin 2\alpha \sin(\beta-\gamma) + \ldots = 0,$$

which, as shown above, is equivalent to

$$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0.$$

2. *The normals at the angular points of a maximum triangle in (and so at the points of contact of a minimum triangle about) an ellipse are concurrent.*

This may be shown without reference to the above *reduced* condition for concurrent normals.

For the condition

$$\sin 2\alpha \sin(\beta-\gamma) + \ldots = 0$$

for concurrent normals follows *at once* from their equations; and at the angular points of a maximum inscribed triangle this condition is satisfied, for at these points we have

$$\beta - \alpha = \gamma - \beta = \frac{2\pi}{3};$$

$$\therefore \quad \gamma - \alpha = \frac{4\pi}{3}, \quad \text{and} \quad \gamma + \alpha = 2\beta.$$

Hence

$$\sin 2\alpha \sin(\beta-\gamma) + \ldots$$

$$= -\sin\frac{\pi}{3}(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)$$

$$= -\sin\frac{\pi}{3}(\sin 2\beta - \sin\overline{\gamma+\alpha}) = 0,$$

and thus the normals are concurrent.

3. *To find the area of the triangle formed by the three normals to an ellipse.*

In the case of the triangle formed by three lines whose equations are of the form $ax+by+c=0$,

$$2\Delta=(a_1b_2c_3)^2/\Pi(a_1b_2)$$
$$=[c_1(a_2b_3-a_3b_2)+\ .\ .\ ]^2/(a_2b_3-a_3b_2)\ .\ .$$

Hence, for the normals whose equations are of the form

$$ax\sin\phi-by\cos\phi=\frac{c^2}{2}\sin2\phi,$$

we have

$$2\Delta=\left[\frac{c^2ab}{2}\{\sin2\alpha\sin(\beta-\gamma)+\ .\ .\ \}\right]^2\Big/a^3b^3\sin(\beta-\gamma)\ .\ .$$
$$=\frac{c^4}{4ab}\{\sin2\alpha\sin(\beta-\gamma)+\ .\ .\ \}^2/\sin(\beta-\gamma)\ .\ .$$

Now use the identity (A), and we get

$$2\Delta=\frac{c^4}{2ab}\tan\frac{\beta-\gamma}{2}\ .\ .\ \{\sin(\beta+\gamma)+\ .\ .\ \}^2,$$

the required expression.

4. *If the normal at the points whose eccentric angles are* α, β, λ, δ *be concurrent, then*

$$\alpha+\beta+\gamma+\delta=(2n+1)\pi.$$

This is merely a question in Plane Trigonometry. For the equation to the normal at ϕ is

$$ax\sec\phi-by\operatorname{cosec}\phi=c^2,$$

which is of the *form* $\quad a\sec\phi-b\operatorname{cosec}\phi=c.$

Denoting $\tan\phi$ by t, we have

$$a\sqrt{1+t^2}-\frac{b}{t}\sqrt{1+t^2}=c$$
$$\therefore\quad(1+t^2)(at-b)^2=c^2t^2,$$

an equation of the fourth degree in t, the roots of which are the tans. of the angles α, β, γ, δ. And since the coefficients of t and t^3 are obviously the same, we have, with the usual notation,

$$\tan(\alpha+\rho+\gamma+\delta)\equiv(s_1-s_3)/(1-s_2+s_4)=0$$
$$\therefore\quad\alpha+\beta+\gamma+\delta=n\pi,\text{ and not }(2n+1)\pi\textit{ necessarily}.$$

Let us now denote $\tan\frac{\phi}{2}$ by t, when we get

$$a.\frac{1+t^2}{1-t^2}-b.\frac{1+t^2}{2t}=c$$

$$\therefore\quad b(t^4-1)+2(a+c)t^3+2(a-c)t=0,$$

where t^2 is absent, and $s_4=-1$, so that $1-s_2+s_4=0$ while s_1-s_3 is not $=0$. Thus

$$\tan\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)=\infty$$

$$\therefore\quad \tfrac{1}{2}(\alpha+\beta+\gamma+\delta)=n\pi+\frac{\pi}{2}$$

and $$\alpha+\beta+\gamma+\delta=(2n+1)\pi,$$

which is the correct result.

The reason why the first biquadratic does not give the exact result being that, to obtain it, we performed the operation of *squaring;* and when we square we must expect, as usual, a result of greater generality than, but not contradictory of, the actual result.

A General Theorem on the Nine-points Circle.

By V. Ramaswami Aiyar, M.A.

Theorem: *If any conic be inscribed in a given triangle and a confocal to it pass through the circumcentre, then the circle through the intersection of these two confocals touches the nine-points circle of the triangle.*

Demonstration: Let X (Fig. 10) be any conic inscribed in the triangle ABC; O, H, N its circumcentre, orthocentre and nine-points centre; let R be the circumradius.

Let X be any conic inscribed in the triangle ABC; P, Q its foci; M its centre; and α, β its semi-axes.

Let Y be a confocal to X passing through the circumcentre O; and let ρ be the radius of the circle through the intersections of X and Y. We have to show that this circle touches the nine-points circle of ABC.

This will be proved if we show that $\rho = \frac{1}{2}\mathrm{R} \pm \mathrm{MN}$. This can be shown with the aid of the following propositions:

Lemma I. The circle passing through the intersections of the confocals

$x^2/a^2 + y^2/b^2 = 1$ and $x^2/(a^2+\lambda) + y^2/(b^2+\lambda) = 1$ is $x^2 + g^2 = a^2 + b^2 + \lambda$;

this circle is the *mutual orthoptic circle* of the two confocals.

Lemma II. If P and Q be the foci of any conic X inscribed in a triangle ABC we have

$$(\mathrm{R}^2 - \mathrm{OP}^2)(\mathrm{R}^2 - \mathrm{OQ}^2) = 4\beta^2\mathrm{R}^2.$$

[Professor Genese, *Educatianal Times*, Q. 10879; for a solution see p. 37, Vol. 57 of the *Mathematical Reprints.*]

Lemma III. Any conic X being inscribed in a triangle ABC its director circle cuts the *polar circle* of the triangle orthogonally. The centre of the polar circle is the orthocentre H and the square of its radius $= -\frac{1}{2}(\mathrm{R}^2 - \mathrm{OH}^2)$.

Now by lemma I. applied to the confocals X and Y we have

$$\rho^2 = \beta^2 + \left(\frac{\mathrm{OP} \pm \mathrm{OQ}}{2}\right)^2$$

$$= \tfrac{1}{4}(\mathrm{OP}^2 + \mathrm{OA}^2 + 4\beta^2) \pm \tfrac{1}{2}\mathrm{OP}\,.\,\mathrm{OQ} \qquad . \quad . \quad (1)$$

Lemma II. gives

$$R^4 - R^2(OP^2 + OQ^2 + 4\beta^2) + OP^2 . OQ^2 = 0 \qquad . \qquad . \quad (2)$$

In (1) and (2) the expression $OP^2 + OQ^2 + 4\beta^2$ occurs; this is readily seen to be equal to $2(\alpha^2 + \beta^2 + OM^2)$ . . . . (3)

Again by lemma III. we have

$$(\alpha^2 + \beta^2) - \tfrac{1}{2}(R^2 - OH^2) = MH^2;$$

$$\therefore \quad \alpha^2 + \beta^2 + OM^2 = OM^2 + MH^2 + \tfrac{1}{2}(R^2 - OH^2)$$

$$= \tfrac{1}{2}R^2 + 2MN^2 \quad . \quad . \quad . \quad . \quad (4)$$

By (3) and (4) we have

$$OP^2 + OQ^2 + 4\beta^2 = R^2 + 4MN^2 \quad . \quad . \quad . \quad (5)$$

Using this in (2) we get a pretty simple result

$$OP . OQ = 2R . MN \quad . \quad . \quad . \quad . \quad (6)$$

Now making use of (5) and (6) in equation (1) we get

$$\rho^2 = \tfrac{1}{4}R^2 + MN^2 \pm R . MN$$

$$= (\tfrac{1}{2}R \pm MN)^2$$

$\therefore \ \rho = \frac{1}{2}R \pm MN$; and the theorem is proved.

Corollary.—A beautiful theorem, due to Mr M'Cay, of which Feuerbach's theorem is a particular case, is itself a particular case of the theorem now given; Mr M'Cay's theorem may be thus stated: "If either axis of a conic inscribed in a given triangle pass through the circumcentre, then the corresponding auxiliary circle of the conic touches the nine-points circle of the triangle." [See *Casey's Conics*, 2nd Edition, p. 329.]

On the Geometical Represetation of Elliptic Integrals of the First Kind.

By Alex. Morgan, M.A., B.Sc.

[*See page* 2 *of present volume.*]

Dr T. B. Sprague, M.A., F.R.S.E., was elected President in room of the Rev. John Wilson, deceased.

Fourth Meeting, 12th February 1896.

T. B. SPRAGUE, Esq., LL.D., President, in the Chair.

The Steady Motion of a Spherical Vortex.

By H. S. CARSLAW.

The possibility of the steady motion of a spherical vortex of constant vorticity in an infinite homogeneous liquid was first pointed out by Hill in the *Phil. Trans.*, 1894, pp. 213–245. He had already discussed a case of motion which had for the surfaces always containing the same particles those given by the equation

$$\varpi^2\left(\frac{\varpi^2}{a^2} + \frac{(z-Z)^2}{c^2} - 1\right) = \text{constant},$$

a particular surface being

$$\varpi^2(\varpi^2 + (z-Z)^2 - a^2) = 0.$$

Since these surfaces are of invariable form it is possible to imagine the fluid limited by any one of them, provided a rigid frictionless boundary having the shape of the limiting surface be supplied and supposed to move parallel to the axis of z with velocity $\dot{Z}$. His previous result gave the velocity components of a possible rotational motion inside this boundary. Further he showed that for the particular case of the surface being a sphere the rotational motion inside is continuous as regards velocity normal and pressure with a certain irrotational motion in all space outside. This irrotational motion is that produced by the sphere moved in the same direction with the same velocity.

Thus in this case the boundary may be removed and the possibility of the state of motion known as Hill's Spherical Vortex is established.

The object of this paper is to show that by using the ordinary hydrodynamical equations, this and other allied types of steady motion not yet noted may be quickly demonstrated. As the problems deal with spheres and spherical shells the equations are taken in spherical coordinates.

§ 1.

Consider the possibility of a spherical vortex of constant vorticity and density ρ_1 moving with velocity V in an infinite homogeneous liquid of density ρ_2. Impressing on the whole an equal and opposite velocity we have the case of the vortex at rest and fluid streaming from infinity with velocity V. This requires the following equations to be satisfied by the current function:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1-\mu^2}{r^2}\quad\frac{\partial^2}{\partial\mu^2}\right)\psi_1 = \mathrm{M}r^2\sin^2\theta, \quad . \quad . \quad *(1)$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1-\mu^2}{r^2}\quad\frac{\partial^2}{\partial\mu^2}\right)\psi_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $$\omega = -\frac{\mathrm{M}}{2}r\sin\theta,$$

$$\psi_2 = -\tfrac{1}{2}\mathrm{V}r^2\sin^2\theta \text{ at infinity}, \quad . \quad . \quad . \quad (3)$$

$$\psi_1 \text{ and } \psi_2 \text{ constant at } r=a; \quad . \quad . \quad . \quad (4)$$

also the pressure must be continuous at $r=a$. . . . . (5)

As in Hill's paper we have $$\psi_1 = \frac{\mathrm{M}}{10}r^2(r^2-a^2)\sin^2\theta, \quad . \quad . \quad (6)$$

$$\psi_2 = -\tfrac{1}{2}\mathrm{V}\left(r^2-\frac{a^3}{r}\right)\sin^2\theta. \quad . \quad (7)$$

We have still to examine the pressure equations.

These may be written $$\frac{p}{\rho_1}+\tfrac{1}{2}q^2-\mathrm{M}\psi_1 = \frac{\mathrm{P}}{\rho_1}+\tfrac{1}{2}\mathrm{U}^2, \quad . \quad . \quad (8)$$

$$\frac{p}{\rho_2}+\tfrac{1}{2}q^2 \qquad = \frac{\Pi}{\rho_2}+\tfrac{1}{2}\mathrm{V}^2, \quad . \quad . \quad (9)$$

when P and Π are the pressures at the centre and at infinity, U the velocity at the centre.

By using the values given for ψ_1 and ψ_2 we have at once for the pressure at $r=a$

$$\frac{p}{\rho_1}+\frac{1}{50}\mathrm{M}^2a^4\sin^2\theta = \frac{\mathrm{P}}{\rho_1}+\frac{1}{50}\mathrm{M}^2a^4,$$

$$\frac{p}{\rho_2}+\frac{9}{8}\mathrm{V}^2\sin^2\theta \quad = \frac{\Pi}{\rho_2}+\tfrac{1}{2}\mathrm{V}^2,$$

* Basset, Hydrodynamics, Vol. II., p. 81, Equations (55) and (57) in spherical coordinates. The axis from which θ is increased is parallel to the direction of V

Therefore we must have $$\mathrm{V} = -\frac{2}{15}\sqrt{\frac{\rho_1}{\rho_2}}\,.\,\mathrm{M}a^2, \quad . \quad . \quad . \quad (10)$$

$$\mathrm{P} = \Pi - \frac{5}{8}\rho_2\mathrm{V}^2. \quad . \quad . \quad . \quad . \quad (11)$$

Further we find for the pressure at the point (r, θ) inside the vortex

$$p + \frac{9}{8a^4}\cdot\rho_2\mathrm{V}^2[(a^2-r^2)^2 + r^2\sin^2\theta(3r^2-2a^2) + 5r^2(a^2-r^2)\sin^2\theta] = \mathrm{P} + \frac{9}{8}\rho_2\mathrm{V}^2.$$

This may be written

$$p = \mathrm{P} - \frac{9}{32}\rho_2\mathrm{V}^2 + \frac{9}{8a^4}\rho_2\mathrm{V}^2\left[\left(r^2 - \frac{a^2}{2}\right) + r^2(3a^2-2r^2)\cos^2\theta\right] \quad . \quad (12)$$

Therefore p is least when $\quad r = \frac{a}{\sqrt{2}}$ and $\theta = \pm\frac{\pi}{2}$,

and this minimum pressure $$= \Pi - \frac{29}{32}\rho_2\mathrm{V}^2 = \mathrm{P} - \frac{9}{32}\rho_2\mathrm{V}^2. \quad . \quad (13)$$

We have thus found that, whether the density inside is the same or different from that outside, the pressure is least at the points $\left(\frac{a}{\sqrt{2}}, \pm\frac{\pi}{2}\right)$: also a hollow will begin to form there if $\Pi < \frac{29}{32}\rho_2\mathrm{V}^2$: while the velocity of translation of the vortex is given by $\mathrm{V} = \frac{2}{15}\sqrt{\frac{\rho_1}{\rho_2}}\,.\,\mathrm{M}a^2$.

In the circular cylindrical vortex and the vortex ring treated of by Hicks we find cases of hollow vortices. It might have been expected that a possible state of steady motion would be a spherical vortex with a concentric hollow. The fact that the points of minimum pressure are as found above dissipates this idea: the same conclusion might be reached by work resembling that in the next portion of this paper.

§ 2.

By means of a similar analysis it can be shown that with certain conditions between the vorticities, densities, and radii, it is possible

to have a vortex in which the liquid is arranged in spherical strata. The work may be generalised, but I take as sufficiently illustrative the case of two strata.

Here we have

$$\psi_3 = -\tfrac{1}{2}V\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\psi_2 = \tfrac{1}{5}M_2 r^4(-I_2 + I_4) + \left(A_2 r^2 + \frac{B_2}{r}\right)I_2 + \left(A_4 r^4 + \frac{B_4}{r^3}\right)I_4, \quad . \quad (15)$$

$$\psi_1 = \tfrac{1}{5}M_1 r^4(-I_2 + I_4) + \quad C_2 r^2 I_2 \quad + \quad C_4 r^4 I_4, \quad . \quad (16)$$

where $\dfrac{dI_n}{d\mu} = P_{n-1}(\mu)$.

These functions are treated of by Sampson in *Philosophical Transactions*, 1890, and, from the tables he gives, we see at once that $I_4 - I_2 = \dfrac{5}{8}\sin^4\theta$. For that reason we do not carry our expressions for ψ past I_4.

We shall have demonstrated the possibility of this case of motion if we can determine the constants to satisfy

$$\psi = \text{ constant at } r = b \text{ and } r = a, \ (b < a); \quad . \quad . \quad (17)$$

$$\text{Pressure continuous at } r = b \text{ and } r = a. \quad . \quad . \quad (18)$$

The results we now give follow at once from the expressions for ψ and those we obtain for the pressure

(*a*). ψ constant at $r = b$ and $r = a$.

For this we have

$$\left.\begin{aligned} \tfrac{1}{5}M_1 b^4 + C_4 b^4 &= 0, \\ -\tfrac{1}{5}M_1 b^4 + C_2 b^2 &= 0, \\ \tfrac{1}{5}M_2 a^4 + A_4 a^4 + \frac{B_4}{a^3} &= 0, \\ -\tfrac{1}{5}M_2 a^4 + A_2 a^2 + \frac{B_2}{a} &= 0, \\ \tfrac{1}{5}M_2 b^4 + A_4 b^4 + \frac{B_4}{b^3} &= 0, \\ -\tfrac{1}{5}M_2 b^4 + A_2 b^2 + \frac{B_2}{b} &= 0. \end{aligned}\right\} \quad (19)$$

(β). The pressure continuous at $r=b$ and $r=a$.

Hence we must have

$$\left.\begin{aligned}
&\tfrac{4}{5}M_2b^3+4A_4b^3-3\frac{B_4}{b^4}=0,\\
&-\tfrac{4}{5}M_2a^2+2A_2-\frac{B_2}{a^3}=3V\sigma_2,\\
&-\tfrac{4}{5}M_2b^2+2A_2-\frac{B_2}{b^3}=-\tfrac{2}{5}M_1b^2\sigma_1{}^{-1},
\end{aligned}\right\} \quad . \quad (20)$$

where $\sigma_2=\sqrt{\dfrac{\rho_3}{\rho_2}}$: $\sigma_1=\sqrt{\dfrac{\rho_2}{\rho_1}}$.

From these equations we have

$$\left.\begin{aligned}
&A_2=\tfrac{1}{5}M_2\frac{a^5-b^5}{a^3-b^3}\ ; && A_4=-\tfrac{1}{5}M_2,\\
&B_2=-\tfrac{1}{5}M_2a^3b^3.\frac{a^2-b^2}{a^3-b^3}; && B_4=0,\\
&C_2=\tfrac{1}{5}M_1b^2\ ; && C_4=-\tfrac{1}{5}M_1,
\end{aligned}\right\} \quad . \quad (21)$$

$$\frac{M_2}{M_1}\sigma_1=2b^2.\frac{a^3-b^3}{5a^3b^2-3a^5-2b^5} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$V\sigma_2=\frac{M_2}{15(a^3-b^3)}(5a^2b^3-2a^5-3b^5) \quad . \quad . \quad . \quad . \quad (23)$$

Equation (23) gives the velocity of translation of this vortex; (22), the necessary relation between M_1, M_2, ρ_1, ρ_2, a and b, that the motion may be possible; (21), the expressions for the current function. Thus we have determined all the circumstances of the motion.

Proof of the theorem that the mid points of the three diagonals of a complete quadrilateral are collinear.

By JOHN DOUGALL, M.A.

The following proof of this theorem assumes only *Euclid*, I. 43, and its converse, with the well-known deductions, "the line joining the mid points of two sides of a triangle is parallel to the third side," and "the mid point of one diagonal of a parallelogram is also the mid point of the other." The proof given by Dr Taylor in his *Conics* which suggested the method, makes use of ratios.

Let ABCD (Fig. 16) be a quadrilateral, AD, BC, produced meeting in E, and AB, DC, produced in F. Through each of the angular points of the figure draw parallels to AB, AD, giving two sets of four parallel lines,

AGBF, HCKL, DMNP, EQRS, in the one set,

and AHDE, GCMQ, BKNR, FLPS, in the other set.

By *Euclid*, I. 43,

$$\square^m AC = \square^m CR, \quad \text{and} \quad \square^m AC = \square^m CP.$$

$$\therefore \quad \square^m CP = \square^m CR, \quad \text{and} \quad \therefore \ \text{CNS is a straight line.}$$

∴ the mid points of AC, AN, AS are collinear,

that is, the mid points of AC, BD, EF are collinear.

Fifth Meeting, Friday, 12*th March* 1897.

C. G. Knott, Esq., D.Sc., F.R.S.E., in the Chair.

On a Proof of the Fundamental Combination Theorem.

By J. B. Clark, M.A.

The following proof of the fundamental Combination Theorem does not appear in any of the current text-books on Algebra. It has the twofold advantage of being exceedingly simple and of being quite independent of the fundamental Permutation Theorem.

Let the n different things be represented by n letters

$$a,\ b,\ c,\ d,\ \ldots\ .$$

We can form a 1-combination in n ways.

We can form a 2-combination by taking any one of the n 1-combinations and along with it any one of the remaining $(n-1)$ letters: this gives $n(n-1)$ combinations, but each combination is formed twice, *e.g.*, ab arises when b is taken with a, and also when a is taken with b.

Hence $${}_nC_2 = n \cdot \frac{n-1}{2}.$$

We can form a 3-combination by taking any one of the $\frac{n(n-1)}{2}$ 2-combinations, and along with it any one of the remaining $(n-2)$ letters: this gives $n \cdot \frac{n-1}{2} \cdot (n-2)$ combinations, but each combination is counted thrice, *e.g.*, abc arises from bc with a, from ca with b, and from ab with c.

Hence $${}_nC_3 = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}.$$

In general we can form an r-combination in

$$n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-\overline{r-2}}{r-1} \cdot n-\overline{r-1}$$

ways, and since each combination is counted r times, we have

$${}_nC_r = \frac{n(n-1)(n-2)\ \ldots\ (n-r+1)}{r!}$$

Maximum and Minimum.

By John Alison, M.A., F.R.S.E.

[Abstract.]

The object of this note was to point out that in using the method of limits to find a geometrical maximum or minimum it is not correct to conduct all the reasoning at the final stage when the limit has been reached, and to call attention to the form of statement which lays stress on the fact that the reasoning should be based on the consideration of the quantities involved while they are yet finite. Examples were given from one or two well-known books for students where the fallacious method of proof is adopted. Two of these examples follow :—

(1.) "The maximum or minimum straight line from a given point to a circle is the normal through the point."

Figure 17.

"Let AP be the minimum line drawn from A, and AQ a consecutive position. Then in the limit AP = AQ, ∴ the triangle APQ is isosceles. And since the angle PAQ is indefinitely small, each of the angles APQ, AQP is ultimately a right angle. And PQ being in the limit the direction of the tangent at P, AP is normal at P."

If "consecutive" means that the lines are coincident, then all the reasoning concerns a triangle which has already vanished and whose properties while it was finite were not examined. If "consecutive" simply means neighbouring, then the same reasoning would prove that any line is normal to a curve. For, instead of "Let AP be the minimum line," read "Let AP be any line drawn from A to the curve and AQ a consecutive position," and so on as before. Indeed the writer in one of the books considered falls into this snare in an equally simple case.

These objections do not apply if we say—If any line AP be taken in the neighbourhood of the minimum line and on one side of it, an equal line AQ can be found on the other side of it. Then APQ is an isosceles triangle, and the bisector of PAQ is perpendicular to the chord PQ. This is true of any such pair of equal lines, and hence is true of the coincident pair at the minimum position; and the bisector which is now coincident with AP and AQ is still perpendicular to PQ which is now a tangent.

(2.) "Of all quadrilaterals which can be formed from four straight lines of given lengths, the maximum is that which can be inscribed in a circle."

FIGURE 18 (*a*).

"Let ABCD be the position of maximum area. Take ABC′D′ a consecutive position keeping AB fixed. Let AD, BC meet in O. Then since AD = AD′,

∴ ultimately the angle ADD′ is a right angle;

∴ also ODD′ is a right angle, and OD = OD′ ultimately.

Similarly	OC = OC′ ultimately.
And	CD = C′D′ ;
∴ angle	DOC = D′OC′,
and triangle	OCD = OC′D′;
∴ angle	DOD′ = COC′.

Again, in the limit the area ABCD = ABC′D′;

∴ triangle OAB = area OC′BAD′.

Taking away the common part OBAD′, we get the triangle OAD′ = OBC′, and an angle AOD′ of the one = BOC′ of the other;

$$\therefore\ OA \cdot OD' = OB \cdot OC';$$

$$\therefore \text{ in the limit } OA \cdot OD = OB \cdot OC;$$

∴ A, B, C, D are concyclic."

Now if this proof had read :—

Let ABCD be *any quadrilateral whatever* formed by the four given lines. Take ABC′D′ a consecutive position keeping AB fixed—and so on as before, we should reach the conclusion that A, B, C, D are concyclic, which is obviously wrong.

Whatever be the fallacy in the second reading of our proof, it is present in the first.

It is asserted that triangle OAD′ = OBC′. But it must be remembered that the whole of the reasoning is being conducted at the final stage of the approach of the one figure to the position of the other and when each of these triangles has become zero; and although there is some circumlocution, the fact that each is zero is the only ground for asserting that their ratio is equal to 1, and it is not a valid ground.

Put shortly, the proof is this :—

FIGURE 18 (*b*).

Let ABCD be the position of maximum area. Then the flat triangle AOD = BOC and their angles at O are equal,

$\therefore$ OA . OD = OB . OC.

$\therefore$ A, B, C, D are concyclic.

It obviously applies to any quadrilateral.

The following is not open to the same objection :—

Let ABC′D′, ABC″D″ be two equal areas on opposite sides of the maximum position.

Bisect D′D″ and C′C″ and let AD, BC meet in O. Then AD′OD″, BC′OC″ are kites having OD′ = OD″, OC′ = OC″, and D′C′ = D″C″.

$\therefore$ triangles OD′C′, OD″C″ are congruent.

$\therefore$ after taking away the common $\angle$ D″OC′, $\angle$ D′OD″ = C′OC″

$\therefore$ their halves $\angle$ AOD″ and $\angle$ BOC′ are equal.

Also, since triangles OD′C′, OD″C″ are congruent and the quadrilaterals are equal,

$\therefore$ OD′ABC′ = OD″ABC″.

Take away OD″ABC′ and the kites are proved equal in area and so are their halves, AOD″ and BOC′.

It follows that OA . OD″ = OB . OC′.

This is true for every such pair of equal quadrilaterals, and therefore for the coincident pair, when D′D″ coincide on OA and C′C″ coincide on OB.

$\therefore$ for the maximum position

OA . OD = OB . OC,

i.e., ABCD is cyclic.

An application of Sturm's Functions.

By J. D. HÖPPNER.

A Geometrical Proof of certain Trigonometrical Formulæ.

By J. W. BUTTERS, M.A., B.Sc.

§1. Direct geometical proofs of the addition theorem, and others allied to it, are often valid for only a limited range of values, and the constructions used are applicable to only one set of formulæ. In the following paper a method is given which is valid for angles of all values and which is applicable to all the usual formulæ involving sines, cosines, and their simple products. One of each set is proved: the others may be obtained either directly or by substitution in the usual manner.

§2. The following construction for $2a$, although not *obviously* valid for all values of a, is inserted as having led to the general construction. It should be compared with that in §4.

FIGURE 19.

$$\begin{aligned}\sin 2a &= \frac{\text{MA}}{r}\\ &= \frac{\text{MA}}{\text{XA}}\cdot\frac{\text{XA}}{r}\\ &= \frac{\text{MA}}{\text{XA}}\cdot\frac{2\text{BA}}{r}\\ &= \cos a\,.\,2\sin a\\ &= 2\sin a\,.\cos a\end{aligned}$$

$$\begin{aligned}\cos 2a = \text{OM}/r = \text{OX}/r - \text{MX}/r &\\ &= 1 - \frac{\text{MX}}{\text{XA}}\cdot\frac{\text{XA}}{r}\\ &= 1 - \sin a\,.\,2\sin a\\ &= 1 - 2\sin^2 a\end{aligned}$$

FIGURE 20.

§3. If AB be a chord subtending an angle θ at the centre of a circle of radius r, and ϕ be the angle between OX and the perpendicular OC to AB, then, since OY, AB are respectively perpendicular to OX, OC, ϕ is the angle between OY and AB.

Hence the projection of AB on OY is $AB\cos\phi$

$$= 2OB\cos\phi$$

$$= 2r\sin\tfrac{1}{2}\theta \,.\, \cos\phi \qquad - \qquad - \quad (1).$$

Also angle between OX and AB is $\left(\dfrac{\pi}{2}+\phi\right)$

$$\therefore \text{ projection of AB on OX is } 2r\sin\tfrac{1}{2}\theta\cos\left(\frac{\pi}{2}+\phi\right)$$

$$= -2r\sin\tfrac{1}{2}\theta\sin\phi \qquad - \qquad - \quad (2).$$

FIGURE 21.

§ 4. Since the projection of OA = the projection of OX plus the projection of XA, if we project these on OX we get from (2)

$$r\cos 2a = r - 2r\sin a \,.\, \sin a$$

$$\text{or } \cos 2a = 1 - 2\sin^2 a.$$

For this operation we shall in future write

Project (OA = OX + XA) on OX

Thus to get $\sin 2a$: project (OA = OX + XA) on OY.

FIGURE 22.

§ 5. Project (OB = OA + AB) on OY

$$r\sin 3a = r\sin a + 2r\sin a\cos 2a \qquad \text{from (1)}$$

$$\text{or } \sin 3a = \sin a + 2\sin a(1 - 2\sin^2 a)$$

$$= 3\sin a - 4\sin^3 a.$$

Similarly $\cos 3a$ by projecting on OX.

The sines and cosines of higher multiples may be obtained in the same manner—the reductions are usually somewhat shorter than by the ordinary methods.

FIGURE 23.

§ 6. Projecting (AB = AO + OB)

or say (OB − OA = AB)

on OY, and dividing by r

$$\sin a - \sin\beta = 2\sin\tfrac{1}{2}(a-\beta)\cos\tfrac{1}{2}(a+\beta).$$

For the other three related formulæ

put $-\beta$ for β; project on OX; put $\pi+\beta$ for β.

FIGURE 24.

§7. Project $(AB = AX + XB)$ on OX

$$-2r\sin(\alpha-\beta)\sin(\alpha+\beta) = 2r\sin^2\beta - 2r\sin^2\alpha$$

$$\therefore \sin(\alpha-\beta)\sin(\alpha+\beta) = \sin^2\alpha - \sin^2\beta.$$

For related formulæ

put $\pi/2-\alpha$ for α; project on OY; put $-\beta$ for β.

FIGURE 25.

§8. Project $(AC = AB + BC)$ on OY

$$2r\sin(\alpha+\beta)\cos(\alpha+\beta+\chi) = 2r\sin\alpha\cos(\alpha+\chi) + 2r\sin\beta\cos(2\alpha+\beta+\chi) \quad \text{(A)}.$$

Put $\alpha+\beta+\chi=0$

Then
$$\sin(\alpha+\beta) = \sin\alpha\cos(-\beta) + \sin\beta\cos\alpha$$
$$= \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

Related formulæ are obtained in the usual manner.

From the general formula (A) we may get an endless variety of results: Thus putting almost at random

$A+B$ for $\alpha+\beta$, $C-D$ for $\alpha+\beta+\chi$, and $A+D$ for α

we have $\alpha = A+D$, $\beta = B-D$, $\chi = C-A-B-D$

whence (A) becomes

$$\sin(A+B)\cos(C-D) = \sin(A+D)\cos(C-B) + \sin(B-D)\cos(A+C).$$

The general formula (A) must evidently be expressible in a form involving the three angles symmetrically: to get this, put

$$\alpha = Y - Z$$
$$\beta = Z - X$$
$$\chi = X - Y + Z$$

and we have

$$\sin(Y-Z)\cos X + \sin(Z-X)\cos Y + \sin(X-Y)\cos Z = 0$$

From this as in Fig. 26 we get the well-known theorems for a pencil of rays

$$\sin bc\cos ad + \sin ca\cos bd + \sin ab\cos cd = 0$$

and $$\sin bc\sin ad + \sin ca\sin bd + \sin ab\sin cd = 0$$

(since the cosines become sines when the projection is on OX).

We may also obtain Ptolemy's Theorem and hence Euler's relation $BC \cdot AD + CA \cdot BD + AB \cdot CD = 0$ for points on a range.

FIGURE 27.

§ 9. Project

$$(A_0A_1 + A_1A_2 + \ . \ . \ . \ + A_{n-1}A_n = A_0A_n) \text{ on OY.}$$

We have

$$2r\sin\frac{\beta}{2}\{\cos a + \cos(a+\beta) + . \ + \cos(a+\overline{n-1}\beta)\} = 2r\sin\frac{n\beta}{2}\cos\left(a + \frac{n-1}{2}\beta\right)$$

$$\therefore \ \cos a + \cos(a+\beta) + \ . \ . \ + \cos(a+\overline{n-1}\beta) = \frac{\sin\frac{n\beta}{2}\cos\left(a+\frac{n-1}{2}\beta\right)}{\sin\frac{\beta}{2}}.$$

By projection on OX the cosines become sines.

From these or by projection of a regular (crossed) n-gon we have

$$\cos\frac{m\pi}{n} + \cos\frac{3m\pi}{n} + \ . \ . \ . \ + \cos\frac{(2n-1)m\pi}{n} = 0$$

where m and n are integers; and similarly for the sines.

Sixth Meeting, April 7th, 1897.

Dr Sprague, President, in the Chair.

Certain Expansions of x^n in Hypergeometric Series.

By Rev. F. H. Jackson, M.A.

In this paper the following expansion will be obtained:

$$
\begin{aligned}
(-1)^{n+1}x^n = {} & \frac{(n)_1}{1!}\left[\frac{(x)_r}{0!r!}+\frac{(n-r)_1(x)_{r+1}}{1!r+1!}+\frac{(n-r)_2(x)_{r+2}}{2!r+2!}+\ \ldots\right] \\
& -\frac{(n)_2}{2!}\left[\frac{(2x)_r}{0!r!}+\frac{(n-r)_1(2x)_{r+1}}{1!r+1!}+\frac{(n-r)_2(2x)_{r+2}}{2!r+2!}+\ \ldots\right] \quad (1) \\
& +\frac{(n)_3}{3!}\left[\frac{(3x)_r}{0!r!}+\frac{(n-r)_1(3x)_{r+1}}{1!r+1!}+\frac{(n-r)_2(3x)_{r+2}}{2!r+2!}+\ \ldots\right] \\
& \qquad \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

in which n and r are positive integers. The Series in the square brackets are Hypergeometric Series with a finite number of terms.

Let $\prod\limits_{r=1}^{r=n}(b+a_r)$ denote the product of the n factors

$$b+a_1 \quad b+a_2 \quad b+a_3 \ \ldots \text{ etc.} \quad b+a_n$$

Then we know

$$\prod_{r=1}^{r=n}(a_r) - n\prod_{r=1}^{r=n}(b+a_r) + \frac{n.n-1}{2!}\prod_{r=1}^{r=n}(2b+a_r) + \;\ldots\; + (-1)^n\prod_{r=1}^{r=n}(nb+a_r) \equiv n!(-b)^n \quad - \quad (2)$$

[*Edin. Math. Socy. Proc.*, Vol. XIII., 1895, p. 115 (4).]

If $a_1 = a$

$a_2 = a - 1$

$\cdots$

$\cdots$

$a_n = a - n + 1$

$$\prod_{r=1}^{r=n}(sb+a_r) = (sb+a)(sb+a-1)\;\ldots\;(sb+a-n+1) = (sb+a)_n$$

And we have the identity

$$(a)_n - n(a+b)_n + \frac{n.n-1}{2!}(a+2b)_n - \;\ldots\; \equiv n!(-b)^n \quad - \quad (3)$$

In most of the subsequent work the series on the left side of (3) will be considered for all values of n

The function $(a)_n$ being $\dfrac{\Pi(a)}{\Pi(a-n)}$ in terms of Gauss's Π Function.

The series (3) when n is a positive integer may be written

$$(a)_n - n(a-b)_n + \frac{n.n-1}{2!}(a-2b)_n - \;\ldots\; \equiv n!b^n \quad - \quad (4)$$

When n is unrestricted let us write

$$(a)_n - n(a-b)_n + \;\ldots\ldots\; = f(n,b) \quad - \quad - \quad (5)$$

Then dividing throughout by $(a)_n$ we obtain

$$1 - n\frac{(a-b)_n}{a_n} + \frac{n.n-1}{2}\;\frac{(a-2b)_n}{(a)_n} - \;\ldots$$

$$+(-1)^r\frac{n.n-1\;\ldots\;n-r+1}{r!}\;\frac{(a-rb)_n}{(a)_n} = \frac{f(nb)}{(a)_n} \quad (6)$$

The following equations show fundamental properties of t
function $(a)_n$

$$(a)_n \times (a-n)_m = (a)_{m+n} = (a)_m \times (a-m)_n$$

from which we obtain

$$\frac{(a-rb)_n}{(a)_n} = \frac{(a-n)_{rb}}{(a)_{rb}} \quad - \quad - \quad - \quad -$$

$$\frac{(a)_{rb}}{(a)_{rb+1}} = \frac{1}{a-rb} \quad - \quad - \quad - \quad - \quad - \ ($$

$$\frac{(a)_{rb}}{(a)_{rb+2}} = \frac{1}{(a-rb)(a-rb-1)} \quad - \quad - \quad -$$

$$\cdots$$
$$\cdots$$

$$\cdots$$
$$\cdots$$

By means of the relation (a). The series (6) may be transform
into

$$1 - n\frac{(a-n)_b}{(a)_b} + \frac{n.n-1}{2!}\ \frac{(a-n)_{2b}}{(a)_{2b}} - \ .\ .\ + (-1)^r\frac{(n)_r}{r!}\ \frac{(a-n)_{rb}}{(a)_{rb}} + \ .\ .$$

$$= \frac{f(nb)}{(a)_n} \quad ($$

For convenience in subsequent work, change $a-n$ to c, then

$$1 - n\frac{(c)_b}{(c+n)_b} + \frac{n.n-1}{2!}\ \frac{(c)_{2b}}{(c+n)_{2b}} - \ .\ .\ + (-1)^r\frac{(n)_r}{r!}\ \frac{(c)_{rb}}{(c+n)_{rb}} + \ .\ .$$

$$= \frac{f(nb)}{(c+n)_n} \quad ($$

Now $\dfrac{(c)_{sb}}{(c+n)_{sb}} = \dfrac{\Pi(c)\Pi(c+n-sb)}{\Pi(c-sb)\Pi(c+n)}$

$$= 1 - \frac{n}{1!}\ \frac{sb}{c+1} + \frac{n.n-1}{2!}\ \frac{sb.sb+1}{c+1.c+2} - \ .\ .\ .\ .$$

(subject to conditions for convergenc

On replacing each term of the series on the left side of (8) by an infinite series we have the expression

$$
\begin{aligned}
1-\frac{(n)_1}{1!}&\left[1-\frac{(n)_1}{1!}\,\frac{b}{c+1}+\frac{(n)_2}{2!}\,\frac{b.b+1}{c+1.c+2}-\;\ldots\;+(-1)^r\frac{(n)_r}{r!}\,\frac{b.b+1\;\ldots\;b+r-1}{c+1.c+2\;\ldots\;c+r}+\;\ldots\right]\\
+\frac{(n)_2}{2!}&\left[1-\frac{(n)_1}{1!}\,\frac{2b}{c+1}+\frac{(n)_2}{2!}\,\frac{2b.2b+1}{c+1.c+2}-\;\ldots\;+(-1)^r\frac{(n)_r}{r!}\,\frac{2b.2b+1\;\ldots\;2b+r-1}{c+1.c+2\;\ldots\;c+r}+\;\ldots\right]\\
&\quad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\
+(-1)^s\frac{(n)_s}{s!}&\left[1-\frac{(n)_1}{1!}\,\frac{sb}{c+1}+\frac{(n)_2}{2!}\,\frac{sb.sb+1}{c+1.c+2}-\;\ldots\;+(-1)^r\frac{(n)_r}{r!}\,\frac{sb.sb+1\;\ldots\;sb+r-1}{c+1.c+2\;\ldots\;c+r}+\;\ldots\right]\\
&\quad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots
\end{aligned}
$$

For convenience denote b by $-\beta$ then

$$(-1)^r b.b+1.b+2\;\ldots\;b+r-1=(\beta)_r$$

and the expression (9) may be written, after splitting up the terms into partial fractions

$$
\begin{aligned}
-\frac{(n)_1}{1!}&\left[1+\frac{(n)_1(\beta)_1}{1!}\left\{\frac{1}{c+1}\right\}+\frac{(n)_2(\beta)_2}{2!}\left\{\frac{1}{c+1}-\frac{1}{c+2}\right\}+.+\frac{(n)_r(\beta)_r}{r!}\left\{\frac{1}{r-1!0!c+1}-\frac{1}{r-2!1!c+2}+.+(-1)^{r-1}\frac{1}{0!r-1!c+r}\right\}+.\right]\\
+\frac{(n)_2}{2!}&\left[1+\frac{(n)_1(2\beta)_1}{1!}\left\{\frac{1}{c+1}\right\}+\frac{(n)_2(2\beta)_2}{2!}\left\{\frac{1}{c+1}-\frac{1}{c+2}\right\}+.+\frac{(n)_r(2\beta)_r}{r!}\left\{\frac{1}{r-1!0!c+1}-\frac{1}{r-2!1!c+2}+.+(-1)^{r-1}\frac{1}{0!r-1!c+r}\right\}+.\right]\\
&\quad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\
(-1)^s\frac{(n)_s}{s!}&\left[\qquad\text{A series similar to the above in terms of } s\beta\qquad\right]\\
&\quad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots
\end{aligned}
\qquad (10)
$$

The expression (9) may now be written

$$P^0 + P'\frac{1}{c+1} + P''\frac{1}{c+2} + \;\ldots\ldots\; + P^{(r)}\frac{1}{c+r} + \;\ldots\ldots$$

where $$P^0 = 1 - \frac{(n)_1}{1!} + \frac{(n)_2}{2!} - \;\ldots\ldots$$

and $$(-1)^r P^{(r)} \equiv \frac{(n)_1}{1!}\left[\frac{(n)_r(\beta)_r}{r!\,0!\,\overline{r-1}!} + \frac{(n)_{r+1}(\beta)_{r+1}}{\overline{r+1}!\,1!\,\overline{r-1}!} + \frac{(n)_{r+2}(\beta)_{r+2}}{\overline{r+2}!\,2!\,\overline{r-1}!} + \right.$$

$$-\frac{(n)_2}{2!}\Big[\qquad \text{Similar series to above in } 2\beta$$

$$+\frac{(n)_3}{3!}\Big[\qquad \text{Similar series in } 3\beta$$

$$\cdots \qquad \cdots \qquad \cdots$$
$$\cdots \qquad \cdots \qquad \cdots$$

By means of the coefficients $P^0 . P' \ldots P^{(r)} \ldots$ we obtain an expansion of b^n

For the series $$1 - n\frac{(c)_b}{(c+n)_b} + \frac{n.\overline{n-1}}{2!}\;\frac{(c)_{2b}}{(c+n)_{2b}} - \;\ldots\ldots$$

has been reduced to the form

$$P^0 + P'\frac{1}{c+1} + \;\ldots\ldots\; + P^{(r)}\frac{1}{c+r} + \;\ldots\ldots$$

in which the coefficients P are functions of n and b only.

When n is a positive integer the series (12) $$\equiv \frac{n!\,b^n}{(c+n)_n}$$

$$\equiv \frac{\Pi(n)\Pi(c)}{\Pi(c+n)}b^n$$

$$\equiv b^n\left[1 - \frac{c}{c+1}\cdot\frac{(n)_1}{1!} + \frac{c}{c+2}\;\frac{(n)_2}{2!} - \;\ldots\; + (-1)^r\frac{c}{c+r}\;\frac{(n)_r}{r!} - \;.\right.$$

$$\equiv b^n\left[1 - \frac{(n)_1}{1!} + \frac{(n)_2}{2!} - \;\ldots\ldots\right.$$

$$+\frac{1}{c+1}\cdot\frac{(n)_1}{1!} - \frac{2}{c+2}\;\frac{(n)_2}{2!} + \;\ldots\; - (-1)^r\;\frac{r}{c+r}\;\frac{(n)_r}{r!} + \;.$$

This series must be identical with (13). Equating the coefficients of

$$\frac{1}{c+1} \quad \frac{1}{c+2} \ldots\ldots$$

we get

$$b^n\frac{(n)_1}{0!} = \mathrm{P}'$$

$$-b^n\frac{(n)_2}{1!} = \mathrm{P}''$$

$$\cdots$$

$$\cdots$$

$$-(-1)^r b^n \frac{(n)_r}{r-1!} = \mathrm{P}^{(r)}$$

$$\cdots$$

$$\cdots$$

Therefore since $b = -\beta$

$$\begin{aligned}
(-1)^{n+1}\beta^n\frac{(n)_r}{r-1!} = \frac{(n)_1}{1!}&\left[\frac{(n)_r(\beta)_r}{r!0!r-1!} + \frac{(n)_{r+1}(\beta)_{r+1}}{r+1!1!r-1!} + \ldots\ldots\right] \\
-\frac{(n)_2}{2!}&\left[\frac{(n)_r(2\beta)_r}{r!0!r-1!} + \frac{(n)_{r+1}(2\beta)_{r+1}}{r+1!1!r-1!} + \ldots\ldots\right] \quad (14) \\
&\cdots \qquad \cdots \qquad \cdots \\
&\cdots \qquad \cdots \qquad \cdots \\
-(-)^s\frac{(n)_s}{s!}&\left[\frac{(n)_r(sb)_r}{r!0!r-1!} + \frac{(n)_{r+1}(s\beta)_{r+1}}{r+1!1!r-1!} + \ldots\ldots\right] \\
&\cdots \qquad \cdots \qquad \cdots \\
&\cdots \qquad \cdots \qquad \cdots
\end{aligned}$$

Removing the factor $\frac{(n)_r}{r-1!}$ which is common to both sides of the above equation we obtain

$$\left.\begin{aligned}
(-1)^{n+1}\beta^n = \frac{(n)_1}{1!}&\left[\frac{(\beta)_r}{0!r!} + \frac{(n-r)(\beta)_{r+1}}{1!r+1!} + \frac{(n-r)_2(\beta)_{r+2}}{2!r+2!} + \ldots\right] \\
-\frac{(n)_2}{2!}&\left[\frac{(2\beta)_r}{0!r!} + \frac{(n-r)(2\beta)_{r+1}}{1!r+1!} + \frac{(n-r)_2(2\beta)_{r+2}}{2!r+2!} + \ldots\right] \\
&\cdots \qquad \cdots \qquad \cdots \\
&\cdots \qquad \cdots \qquad \cdots
\end{aligned}\right\} \quad (15)$$

in which r is any positive integer, this is the same as (1).

Putting $r=1$ we have

$$\begin{aligned}
(-1)^{n+1}\beta^n = \frac{(n_1)}{1!}&\left[\frac{\beta}{0!1!} + \frac{n-1.\beta.\beta-1}{1!2!} + \frac{n-1.n-2.\beta.\beta-1.\beta-2}{2!3!} + \ldots\right] \quad (16) \\
-\frac{(n)_2}{2!}&\left[\frac{2\beta}{0!1!} + \frac{n-1.2\beta.2\beta-1}{1!2!} + \frac{n-1.n-2.2\beta.2\beta-1.2\beta-2}{2!3!} + \ldots\right] \\
+ \quad & \text{Similar series.}
\end{aligned}$$

When n is a positive integer. Expression (16) will consist of n series each containing n terms. Thus if $n=3$ we have

$$\left.\begin{aligned}+\beta^3 = &\frac{3}{1!}\left[\frac{\beta}{0!1!}+2.\frac{\beta.\beta-1}{1!2!}+\frac{2.1.\beta.\beta-1.\beta-2}{2!3!}\right]\\ -&\frac{3.2}{2!}\left[\frac{2\beta}{0!1!}+2.\frac{2\beta.2\beta-1}{1!2!}+\frac{2.1.2\beta.2\beta-1.2\beta-2}{2!3!}\right]\\ +&\frac{3.2.1}{3!}\left[\frac{3\beta}{0!1!}+2.\frac{3\beta.3\beta-1}{1!2!}+\frac{2.1.3\beta.3\beta-1.3\beta-2}{2!3!}\right]\end{aligned}\right\} \quad (17)$$

Other expansions of β^3 may be obtained by putting $r=2$ $r=3$ namely

$$\left.\begin{aligned}+\beta^3 = &\frac{3}{1!}\left[\frac{\beta.\beta-1}{0!2!}+\frac{1.\beta.\beta-1.\beta-2}{1!3!}\right]=\frac{3}{1!}\left[\frac{\beta.\beta-1.\beta-2}{0!3!}\right]\\ -&\frac{3.2}{2!}\left[\frac{2\beta.2\beta-1}{0!2!}+\frac{1.2\beta.2\beta-1.2\beta-2}{1!3!}\right]-\frac{3.2}{2!}\left[\frac{2\beta.2\beta-1.2\beta-2}{0!3!}\right]\\ +&\frac{3.2.1}{3!}\left[\frac{3\beta.3\beta-1}{0!2!}-\frac{1.3\beta.3\beta-1.3\beta-2}{1!3!}\right]+\frac{3.2.1}{3!}\left[\frac{3\beta.3\beta-1.3\beta-2}{0!3!}\right]\end{aligned}\right\} \quad (18)$$

Similarly β^4 may be obtained in 4 different forms
and β^n in n different forms by giving r the values $1.2.3\ldots n$ in the series 16.

The series (15) and (16) are perfectly general in form although we have proved the expansion only when n is a positive integer

If any proof exists that subject to conditions for convergence

$$1-n\frac{(c)_b}{(c+n)_b}+\frac{n.n-1}{2!}\;\frac{(c)_{2b}}{(c+n)_{2b}}-\;\ldots\ldots=b^n\frac{\Pi(n)\Pi(c)}{\Pi(c+n)} \quad . \quad (19)$$

when n is unrestricted, then the expansions (15) (16) *will hold generally subject to convergence,*

The Factorisation of $1-2x^n\cos\alpha+x^{2n}$.

By Professor JACK.

$$\text{Let } S=\sin\theta+x\sin2\theta+x^2\sin3\theta+x^3\sin4\theta+\textit{ ad infinitum}$$

multiply by $2x^n\cos n\theta$

$$\therefore\ S.2x^n\cos n\theta=x^n(\sin\overline{n+1}\theta-\sin\overline{n-1}\theta)+x^{n+1}(\sin\overline{n+2}\theta-\sin\overline{n-2}\theta)$$
$$+x^{n+2}(\sin\overline{n+3}\theta-\sin\overline{n-3}\theta)$$

$$\therefore\ S.2x^n\cos n\theta=S-(\sin\theta+x\sin2\theta+\ .\ .\ .\ +x^{n-1}\sin n\theta)$$
$$+S.x^{2n}-(x^n\sin\overline{n-1}\theta+x^{n+1}\sin\overline{n-2}\theta+\ .\ +x^{2n-2}\sin\theta)$$

Transpose, etc.

$$\therefore\ (1-2x^n\cos n\theta+x^{2n})S=$$
$$\begin{Bmatrix}\sin\theta+x\sin2\theta+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ +x^{n-1}\sin n\theta\\ +x^n\sin\overline{n-1}\theta+x^{n+1}\sin\overline{n-2}\theta+\ .\ .\ .\ +x^{2n-2}\sin\ \theta\end{Bmatrix}$$

Let $n=1$

$\therefore$ $(1-2x\cos\theta+x^2)S=\sin\theta$ (Bracket reduces to one term here.)

DIVIDE.

$$\therefore\ \frac{1-2x^n\cos n\theta+x^{2n}}{1-2x\cos\theta+x^2}=\frac{\{\sin\theta+x\sin2\theta+\ .\ .\ +x^{2n-1}\sin2\theta+x^{2n-2}\sin\theta\}}{\sin\theta.}$$

and when $\cos n\theta$ is given there are n values only of $\cos\theta$

$\therefore$ there are n quadratic factors similar to the above.

The C-Discriminant as an Envelope.

By Mr JAS. A. MACDONALD.

Seventh Meeting, 14th May 1897.

Professor GIBSON in the Chair.

The Bessel Functions and their Zeros.

By Dr PEDDIE.

A Geometrical Theorem with application to the Proof of the Collinearity of the mid-points of the Diagonals of the Complete Quadrilateral.

By R. F. MUIRHEAD, M.A., B.Sc.

Geometrical Note.

By R. TUCKER, M.A.

On the sides BC, CA, AB of the triangle ABC are described two sets of equilateral triangles,
the set $Ba'C$, $Cb'A$, $Ac'B$ externally, and
the set BaC, CbA, AcB internally.
The lines Aa', Bb', Cc' cointersect in Q, the centre of Perspective of the triangles ABC, $a'b'c'$,
and the lines Aa, Bb, Cc in P, the centre of Perspective of ABC, abc.
Since a, a', b, b', c, c' are on the perpendicular bisectors of BC, CA, AB, their joins cointersect in the circumcentre, O, which is the centre of Perspective of abc, $a'b'c'$.

Now $$Oa' = \ \ 2R\cos(60° - A),$$
$$Oa = -2R\cos(60° + A),$$

hence $$aa' = a\sqrt{3}, \quad bb' = b\sqrt{3}, \quad cc' = c\sqrt{3}:$$

and also $$\Sigma(aa')^2 = 3 \quad \Sigma(a^2) = 3k.$$

Using trilinear coordinates,

Q is $$\alpha\sin(60° + A) = \beta\sin(60° + B) = \gamma\sin(60° + C);$$

P is $$\alpha\sin(60° - A) = \beta\sin(60° - B) = \gamma\sin(60° - C).$$

Hence we have the equations to

PQ, $\Sigma a\sin(60^\circ + \mathrm{A})\sin(60^\circ - \mathrm{A})\sin(\mathrm{B} - \mathrm{C}) = 0,$

OQ, $\Sigma a\sin(60^\circ + \mathrm{A})\cos(60^\circ - \mathrm{A})\sin(\mathrm{B} - \mathrm{C}) = 0,$

OP, $\Sigma a\sin(60^\circ - \mathrm{A})\cos(60^\circ + \mathrm{A})\sin(\mathrm{B} - \mathrm{C}) = 0,$

which evidently pass through the symmedian point.

From the Δ Oab we get

$$c'^2 = (ab)^2 = a^2 + b^2 + ab\cos\mathrm{C} - \sqrt{3}\,ab\sin\mathrm{C};$$

$$a'^2 = b^2 + c^2 + bc\cos\mathrm{A} - \sqrt{3}\,bc\sin\mathrm{A};$$

$$b'^2 = c^2 + a^2 + ca\cos\mathrm{B} - \sqrt{3}\,ca\sin\mathrm{B}.$$

Hence $$\Sigma a'^2 = \tfrac{5}{2}\Sigma a^2 - b\sqrt{3}\,\Delta.$$

From the Δ O$a'b'$ we get

$$c''^2 = (a'b')^2 = a^2 + b^2 + ab\cos c + \sqrt{3}\,(2\Delta);\ \text{and so on.}$$

Hence $$\Sigma a''^2 = \tfrac{5}{2}\Sigma a^2 + b\sqrt{3}\,\Delta.$$

also $$a'^2 + a''^2 = 3(b^2 + c^2) - a^2,\ \text{etc.},$$

and $$a''^2 - a'^2 = 4\Delta\sqrt{3} = b''^2 - b'^2 = c''^2 - c'^2.$$

Let Δ', Δ'' be the areas respectively of Oab, O$a'b'$,

then $$2\Delta' = 5\Delta - \sqrt{3}k/4,$$

and $$2\Delta'' = 5\Delta + \sqrt{3}k/4.$$

Hence $$\Delta' + \Delta'' = 5\Delta.$$

If ω', ω'' be the Brocard angles of the triangles

$$\cot\omega' = \frac{5\cot\omega - 3\sqrt{3}}{5 - \sqrt{3}\cot\omega}, \qquad \cot\omega'' = \frac{5\cot\omega + 3\sqrt{3}}{5 + \sqrt{3}\cot\omega}.$$

Again $$(\mathrm{A}a')^2 = c^2 + a^2 - 2ca\cos(60^\circ + \mathrm{B})$$
$$= \frac{\mathrm{K}}{2} + 2\Delta\sqrt{3} = 2\Delta(\cot\omega + \sqrt{3});$$

$$(\mathrm{A}a)^2 = c^2 + a^2 - 2ca\cos(60^\circ - \mathrm{B}) = 2\Delta(\cot\omega - \sqrt{3}):$$

hence $$(\mathrm{A}a')^2 + (\mathrm{A}a)^2 = \mathrm{K} = (\mathrm{B}b')^2 + (\mathrm{B}b)^2 = (\mathrm{C}c')^2 + (\mathrm{C}c)^2.$$

The points a, a' are given by

$$\sin 60^\circ,\quad \sin(\mathrm{C} - 60^\circ),\quad -\sin(60^\circ - \mathrm{B}),$$
$$-\sin 60^\circ,\quad \sin(\mathrm{C} + 60^\circ),\quad \sin(60^\circ + \mathrm{B}),$$

hence the triangles abc, $a'b'c'$ are concentroidal with ABC.

Eighth Meeting, 11th June 1897.

J. B. Clark, Esq., M.A., Vice-President, in the Chair.

Isogonic Centres of a Triangle.

By J. S. Mackay, M.A., LL.D.

Theorem 1

*If on the sides of a triangle ABC equilateral triangles LBC MCA NAB be described externally, AL BM CN are equal and concurrent.**

Figure 28

For triangles BAM NAC are congruent, since two sides and the contained angle in the one are equal to two sides and the contained angle in the other;

therefore BM = NC.

Similarly NC = AL.

Let BM CN meet at V. Join AV LV

Since $\angle$ VBA = $\angle$ VNA

therefore V lies on the circumcircle of ABN

and $\angle$ AVB = 120°

Similarly V lies on the circumcircle of CAM

and $\angle$ CVA = 120°

therefore $\angle$ BVC = 120°

therefore V lies on the circumcircle of BCL

Hence $\angle$ LVC = $\angle$ LBC = 60°,

and AVL is a straight line.

Theorem 2

If on the sides of a triangle ABC equilateral triangles L′BC M′CA N′AB be described internally, AL′ BM′ CN′ are equal and concurrent

* T. S. Davies in the *Gentleman's Diary* for 1830, p. 36.

FIGURE 28

The demonstration of the previous case may be easily modified to suit this one.

Let the point of concurrency be denoted by V′.

The points V V′ are called the *isogonic centres* of ABC.

POSITIONS OF V AND V′

V

When each of the angles A B C is less than 120°, V is inside ABC.

When any one of the angles A B C is greater than 120°, V is outside ABC. For example, if A be greater than 120°, V lies between BA and CA produced.

When ABC is equilateral, its centroid, circumcentre, incentre, orthocentre all coincide, and V coincides with them.

V′

When B is greater than 60° and C is less than 60°, V′ is outside ABC and between CA and CB produced.

When B is less than 60° and C is greater than 60°, V′ is outside ABC and between BA and BC produced.

When B and C are both greater than 60°, or both less than 60°, V′ is outside ABC and between AB and AC produced.

When one of the angles of ABC, for example B, is 60°, V′ coincides with B.

When ABC is equilateral, V′ may be anywhere on the circumcircle of ABC.

Theorem 1 is closely connected with the problem

To find that point the sum of whose distances from the vertices of a triangle is a minimum.

This problem, Viviani relates, was proposed by Fermat to Torricelli, and by him handed over as an exercise to Viviani, who gives * the following construction for obtaining the point.

Let ABC be the triangle, and let each of its angles be less than 120°.

* In the Appendix to his treatise *De Maximis et Minimis*, pp. 144, 150 (1659).

On AB and AC describe segments of circles containing angles of 120°. The arcs of these segments will intersect at the point required.

Viviani's proof that this is the point required is too long to extract.

Thomas Simpson in his *Doctrine and Application of Fluxions*, § 36 (1750) gives the following construction for determining the same point.

Describe on BC a segment of a circle to contain an angle of 120°, and let the whole circle BCQ be completed. From A to Q, the middle of the arc BQC, draw AQ intersecting the circumference of the circle in V, which will be the point required.

In § 431, Simpson treats the more general problem,

Three points A B C being given, to find the position of a fourth point P, so that if lines be drawn from thence to the three former, the sum

$$a \,.\, \mathrm{AP} + b \,.\, \mathrm{BP} + c \,.\, \mathrm{CP}$$

where a b c denote given numbers, shall be a minimum.

Both the particular and the more general problem are discussed † by Nicolas Fuss in his memoir "De Minimis quibusdam geometricis, ope principii statici inventis" read to the Petersburg Academy of Sciences on 25th February 1796. In this memoir, denoting $\mathrm{AV} + \mathrm{BV} + \mathrm{CV}$ by s, Fuss gives the expressions

$$\mathrm{AV} = \tfrac{1}{3}s + \frac{b^2 + c^2 - 2a^2}{3s}$$

$$\mathrm{BV} = \tfrac{1}{3}s + \frac{c^2 + a^2 - 2b^2}{3s}$$

$$\mathrm{CV} = \tfrac{1}{3}s + \frac{a^2 + b^2 - 2c^2}{3s}$$

$$\mathrm{BV} \,.\, \mathrm{CV} + \mathrm{CV} \,.\, \mathrm{AV} + \mathrm{AV} \,.\, \mathrm{BV} = \frac{4\Delta}{\sqrt{3}}$$

$$\mathrm{AV}^2 + \mathrm{BV}^2 + \mathrm{CV}^2 = \frac{a^2 + b^2 + c^2}{2} - \frac{2\Delta}{\sqrt{3}}$$

$$\mathrm{AV} + \mathrm{BV} + \mathrm{CV} = \sqrt{\left(\frac{a^2 + b^2 + c^2}{2} + 2\Delta\sqrt{3}\right)}$$

† See *Nova Acta Academiae . . . Petropolitanae* XI. 235-8 (1798)

The following are some of the properties that may be deduced from the figure consisting of a triangle and the equilateral triangles described on its sides.

(1)* $LV = BV + CV \qquad L'V' = BV' + CV'$

$MV = CV + AV \qquad M'V' = CV' + AV'$

$NV = AV + BV \qquad N'V' = AV' + BV'$

Care must be taken to affix the proper algebraic sign (+ or −) according to the position of V or V′

To prove $\qquad LV = BV + CV.$

Ptolemy's theorem applied to the cyclic quadrilateral VBLC gives

$$LV \cdot BC = BV \cdot LC + CV \cdot LB$$

or
$$LV \cdot BC = BV \cdot BC + CV \cdot BC;$$

therefore
$$LV = BV + CV$$

(2)* $AL = BM = CN = AV + BV + CV$

$AL' = BM' = CN' = AV' + BV' + CV'$

with proper algebraic signs prefixed.

(3)† *The following six triangles are congruent to ABC, and the centres of their circumcircles lie all on the circumcircle of ABC ;*

$$AN'M \quad ANM' \quad NBL' \quad N'BL \quad M'LC \quad ML'C$$

This may be proved by rotating the triangle ABC round A through an angle of 60°, first counterclockwise, and second clockwise; then doing the same thing round B, and round C.

(4)‡ *The internal equilateral triangle described on any side cuts a side of each of the external equilateral triangles on the circumcircle of ABC.*

Let AM′ meet BL at D.

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* W. S. B. Woolhouse in the *Lady's and Gentleman's Diary* for 1865, p. 81. His proof of (1) is different from that in the text.

(1) and (2) are said to be given by Heinen, *Ueber Systeme von Kräften* (1834).

† John Turnbull in the *Lady's and Gentleman's Diary* for 1865, p. 78.

‡ Rev. William Mason and Thomas Dobson in the *Lady's and Gentleman's Diary* for 1865, pp. 76, 78.

Since $\angle$ CAD and $\angle$ CBD are each 60°,
therefore A B C D are concyclic *

(5)† $$AL^2 + AL'^2 = BM^2 + BM'^2 = CN^2 + CN'^2$$
$$= a^2 + b^2 + c^2$$

Join LL'.

Then BC LL' bisect each other perpendicularly in A';
therefore
$$AL^2 + AL'^2 = 2AA'^2 + 2A'L^2$$
$$= 2AA'^2 + 6A'B^2$$
$$= 2AA'^2 + 2A'B^2 + 4A'B^2$$
$$= AB^2 + AC^2 + BC^2$$

(6)† $$AL^2 - AL'^2 = BM^2 - BM'^2 = CN^2 - CN'^2$$
$$= 4\sqrt{3}\,\Delta = \frac{abc\sqrt{3}}{R}$$

Draw AK perpendicular to LL'.

Then
$$AL^2 - AL'^2 = LK^2 - L'K^2$$
$$= (LK + L'K)(LK - L'K)$$
$$= 2A'L \,.\, 2A'K$$
$$= 2\sqrt{3}\,A'B \,.\, 2A'K$$
$$= 4\sqrt{3}\,\Delta = \frac{abc\sqrt{3}}{R}$$

(7)‡ $$AL^2 = \tfrac{1}{2}\left(a^2 + b^2 + c^2 + \frac{abc\sqrt{3}}{R}\right)$$
$$AL'^2 = \tfrac{1}{2}\left(a^2 + b^2 + c^2 - \frac{abc\sqrt{3}}{R}\right)$$

(8)§ $$AL^2 = a^2 + b^2 - 2ab\cos(C + 60°)$$
$$AL'^2 = a^2 + b^2 - 2ab\cos(C - 60°)$$

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* D will be used for a different point in (9).

† Rev. William Mason in the *Lady's and Gentleman's Diary* for p. 75. His proof is different from that in the text.

‡ Rev. William Mason in the *Lady's and Gentleman's Diary* for 1865,

§ The first of these expressions is given in T. S. Davies's editic Hutton's *Course of Mathematics*, I. 470 (1841). It is said to occur al Heinen, *Ueber Systeme von Kräften* (1834)

(9)* *If* AL BM CN *meet* BC CA AB *at* D E F

then $$\frac{1}{VD}+\frac{1}{VE}+\frac{1}{VF}=\frac{2}{AV}+\frac{2}{BV}+\frac{2}{CV}$$

For triangles BVD LVC are similar;

therefore $\quad BV : VD = LV : CV$;

therefore $\quad BV . CV = VD . LV$

$\quad = VD(BV + CV)$;

therefore $$\frac{1}{VD} = \frac{1}{BV}+\frac{1}{CV}.$$

Similarly $$\frac{1}{VE} = \frac{1}{CV}+\frac{1}{AV}$$

and $$\frac{1}{VF} = \frac{1}{AV}+\frac{1}{BV};$$

whence the required result follows.

A corresponding result is true for the point V′ but care must be taken to prefix the proper signs.

(10)† $$AL^2 . AL'^2 = (a^2-b^2)(a^2-c^2)+(b^2-c^2)^2$$

with corresponding values for $BM^2 . BM'^2$ $CN^2 . CN'^2$.

From (7) there is obtained

$$AL^2 . AL'^2 = \left(\frac{a^2+b^2+c^2}{2}\right)^2 - \tfrac{3}{4} . \frac{a^2b^2c^2}{R^2}$$
$$= \tfrac{1}{4}(\ \ a^4+b^4+c^4+2b^2c^2+2c^2a^2+2a^2b^2)$$
$$- \tfrac{3}{4}(-a^4-b^4-c^4+2b^2c^2+2c^2a^2+2a^2b^2)$$
$$= a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2$$

(11)‡ $$AL . AV = \frac{b^2+c^2-a^2}{2}+\frac{abc}{2R\sqrt{3}}$$
$$AL' . AV' = \frac{b^2+c^2-a^2}{2}+\frac{abc}{2R\sqrt{3}}$$

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* W. H. Levy in the *Lady's and Gentleman's Diary* for 1855, p. 71

† W. S. B. Woolhouse in the *Lady's and Gentleman's Diary* for 1865, p. 81.

‡ (11)—(14). Rev. William Mason in the *Lady's and Gentleman's Diary* for 1865, pp. 74, 75.

For $$AV = AB\frac{\sin ABM}{\sin 120^\circ} = \frac{2c}{\sqrt{3}}\sin(A + 60^\circ)\frac{b}{BM}$$

$$AV' = AB\frac{\sin ABM'}{\sin 60^\circ} = \frac{2c}{\sqrt{3}}\sin(A - 60^\circ)\frac{b}{BM'};$$

and $$AL = BM \quad AL' = BM';$$

whence, after making the necessary substitutions, the results follow.

(12) $$AL^2 . AL'^2 . VV'^2 = 3a^2b^2c^2 - \frac{a^2b^2c^2}{3R^2}(a^2 + b^2 + c^2)$$

For $$-2AL . AL'\cos VAV' = AL^2 + AL'^2 - LL'^2$$
$$= a^2 + b^2 + c^2 - 3a^2$$
$$= b^2 + c^2 - 2a^2;$$

therefore $$AL^2 . AL'^2 . VV'^2$$
$$= (AL . AV)^2 AL'^2 + (AL' . AV')^2 AL^2 + (AL . AV)(AL' . AV')(b^2 + c^2 + 2a^2)$$
$$= 3a^2b^2c^2 - \frac{a^2b^2c^2}{3R^2}(a^2 + b^2 + c^2)$$

(13) $$AL^2 . OV^2 = R^2\,\frac{a^2 + b^2 + c^2}{2} - \frac{a^2b^2c^2}{6R^2} + \frac{R\sqrt{3}}{2abc}AL^2 . AL'^2 . VV'^2$$

$$AL'^2 . OV'^2 = R^2\,\frac{a^2 + b^2 + c^2}{2} - \frac{a^2b^2c^2}{6R^2} - \frac{R\sqrt{3}}{2abc}AL^2 . AL'^2 . VV'^2$$

The investigation is too long to be inserted here; the initial steps are

$$2AL . AO\cos OAV = AL^2 + AO^2 - OL^2$$
$$= AL^2 + R^2 - \left(R\cos A + \frac{\sqrt{3}}{2}a\right)^2;$$

and $$AL^2 . OV^2 = AL^2(AO^2 + AV^2 - 2AO . AV\cos OAV);$$

whence, by substitution, the first result follows.

(14) $$OV^2 + OV'^2 + VV'^2 = 2R^2$$

Multiply the first equality in (13) by AL'^2

„ second „ (13) „ AL^2;

add, and make use of the equalities in (5), (6);

then $$AL^2 . AL'^2(OV^2 + OV'^2 + VV'^2)$$
$$= \tfrac{1}{2}R^2(a^2 + b^2 + c^2) - \tfrac{3}{2}a^2b^2c^2$$
$$= 2R^2 . AL^2 . AL'^2$$

(15)* $AV^2 = \dfrac{b^2+c^2-a^2}{3} + \dfrac{(a^2-b^2)(a^2-c^2)}{(a^2-b^2)(a^2-c^2)+(b^2-c^2)^2} \cdot \dfrac{AL'^2}{3}$

$$AV'^2 = \frac{b^2+c^2-a^2}{3} + \frac{(a^2-b^2)(a^2-c^2)}{(a^2-b^2)(a^2-c^2)+(b^2-c^2)^2} \cdot \frac{AL^2}{3}$$

These are obtained from (10) and (11)

(16) $$AV^2 + BV^2 + CV^2 = \frac{a^2+b^2+c^2}{3} + \frac{AL'^2}{3}$$
$$= \frac{a^2+b^2+c^2}{2} - \frac{abc}{2R\sqrt{3}}$$
$$AV'^2 + BV'^2 + CV'^2 = \frac{a^2+b^2+c^2}{3} + \frac{AL^2}{3}$$
$$= \frac{a^2+b^2+c^2}{2} + \frac{abc}{2R\sqrt{3}}$$

For $$(a^2-b^2)(a^2-c^2) + b^2-c^2)(b^2-a^2) + (c^2-a^2)(c^2-b^2)$$
$$= (a^2-b^2)(a^2-c^2) + (b^2-c^2)^2$$

(17) $$\Sigma(AV^2) + \Sigma(AV'^2) = a^2+b^2+c^2$$

(18) *If* R S T *be the circumcentres of* LBC MCA NAB *and* R' S' T' *be the circumcentres of* $L'BC$ $M'CA$ $N'AB$; *then* LR' MS' NT' *are diameters of the first triad of circles and* $L'R$ $M'S$ $N'T$ *are diameters of the second triad of circles.*

For $\angle BR'C = 120°$;
therefore R′ is on the circumcircle of LBC;
and LR′ bisects BC perpendicularly.

(19)† *The triangles* RST $R'S'T'$ *are equilateral.*

Since AV BV CV make with each other angles of 120°,
and ST TR RS are respectively perpendicular to them,
therefore ST TR RS make with each other angles of 60°,
and therefore form an equilateral triangle.‡

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* (15)—(17). W. S. B. Woolhouse in the *Lady's and Gentleman's Diary* for 1865, pp. 84, 82. See the reference to Fuss on p. 102.

† Dr Rutherford in the *Ladies' Diary* for 1825, p. 47. Probably, however, the theorem dates farther back.

‡ Prof. Uhlich ascribes this method to Kunze.

Similarly for R′S′T′.

Or thus: *

If AT, AS be joined,

$$\angle \text{TAS} = 60° + \angle \text{BAC} = \angle \text{NAC}.$$

Now AT AS are corresponding lines in the similar triangles ANB ACM,

therefore $\text{AN} : \text{AC} = \text{AT} : \text{AS}$;

therefore triangles CAN SAT are similar;

therefore
$$\text{CN} : \text{ST} = \text{CA} : \text{SA}$$
$$= \sqrt{3} : 1;$$

therefore $\sqrt{3}\,\text{ST} = \text{CN}$.

Values equal to this are in like manner found for TR RS;
therefore triangle RST is equilateral.

Similarly $\sqrt{3}\,\text{S}'\text{T}' = \text{CN}'$ and R′S′T′ is equilateral.

(20)† *The sum of the squares of the sides of triangles RST R′S′T′ is equal to* $a^2 + b^2 + c^2$.

For
$$3(\text{ST}^2 + \text{S}'\text{T}'^2) = \text{CN}^2 + \text{CN}'^2$$
$$= a^2 + b^2 + c^2;$$

therefore
$$3(\text{ST}^2 + \text{S}'\text{T}'^2 + \text{TR}^2 + \text{T}'\text{R}'^2 + \text{RS}^2 + \text{R}'\text{S}'^2)$$
$$= 3(a^2 + b^2 + c^2)$$

(21) ‡ *The difference of the areas of triangles RST R′S′T′ is equal to the area of ABC; and the sum of the areas of RST R′S′T′ is the arithmetic mean of the three equilateral triangles on BC CA AB.*

For
$$\text{RST} = \text{ST}^2 \frac{\sqrt{3}}{4}$$
$$\text{R}'\text{S}'\text{T}' = \text{S}'\text{T}'^2 \frac{\sqrt{3}}{4}$$

* This is substantially the mode of proof given in the *Ladies' Diary* for 1826, p. 38.

† Dr John Casey. See his *Euclid*, p. 264 (2nd ed., 1884)

‡ Ascribed by Professor Uhlich to Féaux, Arnsberg Programm, p. 4 (1873).

therefore
$$\text{RST}-\text{R}'\text{S}'\text{T}' = (\text{ST}^2-\text{S}'\text{T}'^2)\frac{\sqrt{3}}{4}$$
$$=\tfrac{1}{3}(\text{CN}^2-\text{CN}'^2)\frac{\sqrt{3}}{4}$$
$$=\tfrac{1}{3}\,.\,4\sqrt{3}\,\Delta\,.\,\frac{\sqrt{3}}{4}.$$
$$\text{RST}+\text{R}'\text{S}'\text{T}' = \tfrac{1}{3}(\text{CN}^2+\text{CN}'^2)\frac{\sqrt{3}}{4}$$
$$=\tfrac{1}{3}(a^2+b^2+c^2)\,\frac{\sqrt{3}}{4}.$$

(22)* *Triangles RST R′S′T′ are homologous, and O the circumcentre of ABC is their centre of homology.*

For RR′ bisects BC perpendicularly;
therefore RR′ passes through O.
Similarly for SS′ and TT′.

(23)† *The equilateral triangles RST R′S′T′ have the same centroid as ABC, and their circumcircles pass respectively through V′ and V.*

The perpendiculars from R S T on BC are
$$-\frac{a}{\sqrt{3}} \qquad \frac{\text{AX}}{2}+\frac{b\cos\text{C}}{\sqrt{3}} \qquad \frac{\text{AX}}{2}+\frac{c\cos\text{B}}{\sqrt{3}}$$
where AX is the perpendicular from A to BC.
Now
$$b\cos\text{C}+c\cos\text{B}=a;$$
therefore the sum of these perpendiculars is equal to AX.
Hence the perpendicular on BC from the centroid of RST is equal to $\tfrac{1}{3}$AX;
and similarly for the other perpendiculars.
The centroid therefore of RST, and in like manner of R′S′T′, is the centroid of ABC

* Stated by Reuschle in Schlömilch's *Zeitschrift*, xi. 482 (1866).

† (23)—(29) W. S. B. Woolhouse in the *Lady's and Gentleman's Diary* for 1865, pp. 86, 83, 84.

Again $\angle AV'B = 60°$;

therefore V′ lies on the circumcircle of N′AB.
Now N′T is a diameter of this circle;

therefore $\angle N'V'T = 90°$.

Similarly $\angle M'V'S = 90°$;

therefore
$$\angle SV'T = \angle M'V'N'$$
$$= \angle BV'C$$
$$= \angle SRT;$$

therefore V′ lies on the circumcircle of RST.

(24) *If G be the centroid of ABC then*
$$GV = \tfrac{1}{3}AL' \quad GV' = \tfrac{1}{3}AL.$$

For
$$3GV^2 = (AV^2 + BV^2 + CV^2) - (AG^2 + BG^2 + CG^2)$$
$$= \left(\frac{a^2+b^2+c^2}{2} - \frac{abc}{2R\sqrt{3}}\right) - \frac{a^2+b^2+c^2}{3}$$
$$= \frac{a^2+b^2+c^2}{6} - \frac{abc}{2R\sqrt{3}} = \frac{AL'^2}{3}$$

(25) $$GO^2 + GV^2 + GV'^2 = R^2$$

This follows from (5)

(26) *GV GV′ are the radii of the circumcircles of R′S′T′ RST*

For by (19)

$$ST = \frac{CN}{\sqrt{3}} = \frac{AL}{\sqrt{3}} = \frac{3GV'}{\sqrt{3}} = \sqrt{3}GV'$$
$$S'T' = \frac{CN'}{\sqrt{3}} = \frac{AL'}{\sqrt{3}} = \frac{3GV}{\sqrt{3}} = \sqrt{3}GV$$

(27) *If G′ be the centroid of the triangle OVV′*
$$GG' = \tfrac{1}{3}R$$

For
$$3GG'^2 = (GO^2 + GV^2 + GV'^2) - (G'O^2 + G'V^2 + G'V'^2)$$
$$= R^2 - \tfrac{1}{3}(OV^2 + OV'^2 + VV'^2)$$
$$= R^2 - \tfrac{2}{3}R^2 = \tfrac{1}{3}R^2$$

(28) $$VV' = \left(\frac{GV'}{GV} - \frac{GV}{GV'}\right) . GO$$

For $$AL^2 . AL'^2 . VV'^2 = \frac{3a^2b^2c^2}{R}\left(R^2 - \frac{a^2+b^2+c^2}{9}\right)$$

$$= (AL^2 - AL'^2)^2 . GO^2$$

therefore $$VV' = \frac{AL^2 - AL'^2}{AL \ . \ AL'} . GO$$

$$= \left(\frac{GV'}{GV} - \frac{GV}{GV'}\right) . GO$$

(29) $$OV^2 = 2(GO^2 + GV^2) - GO^2 . \frac{GV^2}{GV'^2}$$

$$OV'^2 - 2(GO^2 + GV'^2) - GO^2 . \frac{GV'^2}{GV^2}$$

From the expression for $AL^2 . OV^2$ in (13) there may be obtained, by substitution and simplification,

$$OV^2 = R^2 - \frac{abc}{3R\sqrt{3}} + \frac{R\sqrt{3}}{3abc} AL'^2 . VV'^2$$

$$= R^2 - (GV'^2 - GV^2) + \frac{GV^2}{GV'^2 - GV^2} . VV'^2$$

$$= R^2 + GV^2 - GV'^2 + GO^2 . \frac{GV'^2 - GV^2}{GV'^2}$$

which by (28) reduces to the required form

(30)* *The area of the triangle OVV' is*

$$\frac{1}{\sqrt{3}} . \frac{(b^2-c^2)(c^2-a^2)(a^2-b^2)}{(b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2}$$

The investigation of this is too long to be given here.

(31)† *If H be the orthocentre of ABC, and Q be the mid-point of HG, then*

$$QV = GO . \frac{GV}{GV'}, \quad QV' = GO . \frac{GV'}{GV}$$

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* Mr Stephen Watson in the *Lady's and Gentleman's Diary* for 1865, p. 78

† (31—(36). W. S. B. Woolhouse in the *Lady's and Gentleman's Diary* for 1865, pp. 84, 85

FIGURE 29

The points H G O are collinear, and $HG = 2GO$

therefore $$HQ = GQ = GO.$$

Now $$OV^2 = 2(GO^2 + GV^2) - QV^2$$
$$OV'^2 = 2(GO^2 + GV'^2) - QV'^2$$

whence, by (29), the results follow

(32) *The points Q V V' are collinear*

and $$QV \cdot QV' = GO^2 = GQ^2$$

For $VV' = QV' - QV$ by (28)

(33) *The bisectors of the angles VGV' VHV' meet VV' at the same point*

Since $$QV : QG = QG : QV'$$

therefore triangles VQG GQV′ are similar;
therefore, if Gn bisect $\angle VGV'$,

$$\begin{aligned} \angle QGn &= \angle QGV + \angle VGn \\ &= \angle QV'G + \angle V'Gn \\ &= \angle QnG; \end{aligned}$$

therefore $$Qn = QG.$$

But since $$QV : QH = QH : QV'$$
therefore triangles VQH HQV′ are similar.
Hence the bisector of $\angle VHV'$ will meet VV' at a point n such that $Qn = QH$.

(34) $$\angle GVH + \angle GV'H = 180^\circ$$

(35) *If with the three points H V V' the parallelogram $HVH'V'$ be completed, then $OH' = R$.*

Denote the mid-point of QG or HO by m' and draw HH′ the diagonal of the parallelogram meeting VV′ in m.

Then $OH'^2 = 4m'm^2$

$$= 2Gm^2 + 2Qm^2 - QG^2$$
$$= 2Gm^2 + 2Vm^2 + 2(Qm - Vm)(Qm + Vm) - QG$$
$$= GV^2 + GV'^2 + 2QV \,.\, QV' - QG^2$$
$$= GV^2 + GV'^2 + QG^2$$
$$= GV^2 + GV'^2 + GO^2 = R^2$$

therefore H′ lies on the circumcircle of ABC.

The position of H′ is further determined by

$$GH' = 2Qm = QV + QV' = GO \,.\, \left(\frac{GV}{GV'} + \frac{GV'}{GV}\right)$$

(36) *If GH' be produced to meet the circle again in H''*

then
$$GH'' = \frac{GV.GV'}{GO}$$

For
$$GH' \,.\, GH'' = R^2 - GO^2$$
$$= GV^2 + GV'^2$$

therefore
$$GH'' = \frac{GV^2 + GV'^2}{GH'}$$
$$= \frac{GV \,.\, GV'}{GO}$$

(37)* *If the squares of the sides of ABC be in arithmetical progression, then AV BV CV are in arithmetical progression*

Denote AV BV CV by x y z; then

$$a^2 = y^2 + z^2 - 2yz \cos 120^\circ = y^2 + z^2 + yz \qquad (1)$$
$$b^2 = z^2 + x^2 - 2zx \cos 120^\circ = z^2 + x^2 + zx \qquad (2)$$
$$c^2 = x^2 + y^2 - 2xy \cos 120^\circ = x^2 + y^2 + xy \qquad (3)$$

Now since $a^2 + c^2 = 2b^2$, therefore

$$x^2 + 2y^2 + z^2 + xy + yz = 2(z^2 + x^2 + zx)$$

or
$$y^2 + \tfrac{1}{2}(z + x)\,y - \tfrac{1}{2}(z + x)^2 = 0$$

The two roots of this quadratic in y are

$$\tfrac{1}{2}(z + x) \quad \text{and} \quad -(z + x)$$

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* Thomas Weddle in the *Mathematician* III. 111 (1848). The solution is taken from p. 165.

If the second be rejected, $y = \frac{1}{2}(z + x)$
and $x\ y\ z$ are in arithmetical progression.

The second root $-(z+x)$ is rejected because it is inconsistent with (1) and (3) unless the triangle be equilateral.
For, from (1) and (3)

$$(z - x)(x + y + z) = a^2 - c^2$$

Now if $\quad y = -(z+x) \quad$ or $\quad x + y + z = 0,$

$$a = c \quad \text{and therefore} \quad a = b = c.$$

When however the triangle is equilateral, not only is $y = \frac{1}{2}(z + x)$ admissible, but likewise $x + y + z = 0$; only in the latter case it is not the three lines $x\ y\ z$ that make equal angles with each other, but two of them and the third produced. In fact, the condition

$$x + y + z = 0$$

expresses the well-known theorem :

If lines be drawn from the vertices of an equilateral triangle to any point in the circumference of the circumcircle, the sum of two of these is equal to the third.

(38) *Triangles ABC RST are homologous; and so are triangles ABC R'S'T'.*

(39) *If the circles VBC VCA VAB be described and through A B C perpendiculars be drawn to VA VB VC, these perpendiculars will form a triangle DEF whose vertices will be situated respectively on the three circles.*

FIGURE 30

(40) *Triangles DEF RST are similar and similarly situated, and V is their centre of similitude.*

For $\quad \angle \mathrm{VBU} = 90°$;
therefore VU is a diameter of the circle LBC,
and consequently passes through R.

(41) *Triangle DEF = four times RST.*

(42) *AL BM CN are equal to the perpendiculars of the triangle DEF*

For $$\angle \text{VLD} = 90^\circ = \angle \text{VAE};$$
therefore DL is parallel to EF.

(43)* *The point V is such that the sum of its distances from the vertices A B C is a minimum.*

Take any point P inside DEF, and let P_1 P_2 P_3 be its projections on the sides of DEF.

Then $$PP_1 + PP_2 + PP_3 = AL = VA + VB + VC.$$
But $$PP_1 + PP_2 + PP_3 < PA + PB + PC$$

(44) *Triangle DEF is the maximum equilateral triangle that can be circumscribed about ABC.*

The problem

About a given triangle to circumscribe the maximum equilateral triangle

was proposed by Thomas Moss in the *Ladies' Diary* for 1755 under the form

In the three sides of an equiangular field stand three trees at the distances of 10, 12, and 16 chains from one another; to find the content of the field, it being the greatest the data will admit of.

The solution given next year was

FIGURE 30

On AB, AC describe segments of circles to contain angles of 60°. Join their centres T S and through A draw EF parallel to ST. Then EC FB will meet at D and form the required triangle DEF.

In Gergonne's *Annales de Mathématiques* I. 384 (1811) there were proposed the two problems:

In any given triangle to inscribe an equilateral triangle which shall be the smallest possible

* The proof given here will be found in Steiner's *Gesammelte Werke* II. 729 (1882)

About any given triangle to circumscribe an equilateral triangle which shall be the greatest possible

It was also suggested that instead of supposing the inscribed and circumscribed triangles to be equilateral they may be supposed similar to given triangles. In this more general form the problems were solved by Rochat, Vecten and others in Vol. II. pp. 88–93 (1811 and 1812)

Their solutions were preceded by the following lemma.

Two triangles t and t' are given in species, and two other triangles T and T′ respectively similar to them are inscribed the one in the other, T′ in T for example. If T′ is the smallest of the triangles similar to t' which it is possible to inscribe in T, the triangle T will be the greatest of the triangles similar to t which it is possible to circumscribe about T′, and conversely.

FIGURE 31

Let ABC be a triangle similar to t, and let DEF be the smallest of all the triangles similar to t' which it is possible to inscribe in it. If ABC is not the greatest of the triangles similar to t which can be circumscribed about DEF, let A′B′C′ greater than ABC be such a triangle. Divide the sides of ABC at D′ E′ F′ as the sides of A′B′C′ are divided at D E F and form the triangle D′E′F′.

Then $\quad ABC : A'B'C' = D'E'F' : DEF.$

If therefore ABC be less than A′B′C′, the triangle D′E′F′ will be less than DEF, which is contrary to the hypothesis.

To prove the converse.

Let ABC be the greatest of the triangles similar to t which it is possible to circumscribe about DEF. If DEF is not the smallest of all the triangles similar to t' which it is possible to inscribe in ABC, let D′E′F′ smaller than DEF be such a triangle. Through D E F let there be drawn three straight lines B′C′ C′A′ A′B′ making with the sides of DEF the same angles that BC CA AB make with their homologues in D′E′F′.

Then $\quad DEF : D'E'F' = A'B'C' : ABC$

If therefore D′E′F′ be less than DEF, the triangle ABC will be less than A′B′C′, which is contrary to the hypothesis.

Hence the following solutions:

About ABC to circumscribe a triangle similar to def *and which shall be the greatest possible.*

FIGURE 32

On CA CB describe externally segments CEA CDB containing angles equal to *e* *d*; and let the arcs of these segments cut each other at P. Through C draw DE perpendicular to PC.

If DB EA meet at F, then DEF is the required triangle.

In ABC to inscribe a triangle similar to def, *and which shall be the smallest possible.*

FIGURE 33

About the triangle *def* circumscribe a triangle *abc* similar to ABC and the greatest possible.

Cut the sides of ABC at D E F in the same manner as those of *abc* are at *d* *e* *f*; then DEF is the required triangle.

Rochat remarks that each of these problems would in general admit of six solutions, unless it is specified beforehand to which sides of the triangle given in species the sides of the circumscribed triangle are to correspond, or to which angles of the triangle given in species the angles of the inscribed triangle are to correspond.

The two preceding problems are discussed in Lhuilier's *Élémens d'analyse géométrique*, pp. 252–5 (1809); and it may be interesting to compare the 26th lemma of the first book of Newton's *Philosophiae Naturalis Principia Mathematica* (2nd ed., 1713), which is

Trianguli specie et magnitudine dati tres angulos ad rectas totidem positione datas, quae non sunt omnes parallelae, singulos ad singulas ponere.

Newton adds as a corollary

Hinc recta duci potest cujus partes longitudine datae rectis tribus positione datis interjacebunt.

The preceding pages contain the early history of the isogonic points, as well as certain properties of them which are not well known either in this country or abroad. Recent researches on the

triangle have brought several of these properties to light again, and have added a considerable number of new ones. Had time and space permitted these latter might have been stated if not proved. Room can be found only for the following references.

Mathesis, II. 187–188 (1882); VI. 211–213 (1886);
VII. 208–220 (1887); IX. 188–189 (1889);
XV. 153–155 (1895)

The article in vol. VII. is by Mr W. S. M'Cay, and contains a note on p. 216 by Prof. Neuberg; that in vol. IX. is by Prof. H. Van Aubel; that in vol. XV. is by Mr H. Mandart.

Mr De Longchamps' *Journal de Mathématiques Élémentaires*,
3rd series, I. 232–236 (1887); III. 99–102, 123–126, 152–154, 180–182, 198–201, 242–245 (1889)
4th series, I. 179–183, 230–233, 248-258, 272–278 (1892)
II. 3–7, 25–29, 49–54, 76–79 (1893)

The article in the volume for 1887 is due to Messrs J. Koehler and J. Chapron; those in the volumes for 1889 and 1892 are due to Mr A. Boutin; those in the volume for 1893 to Mr Bernès

Mr J. M. J. Sachse's *Der fünfte merkwürdige Punkt im Dreieck* (Coblenz, 1875)

Dr Heinrich Lieber's *Ueber die isogonischen und isodynamischen Punkte des Dreiecks* (Programm der Friedrich-Wilhelms-Schule zu Stettin, 1896)

Hoffmann's *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, XXVIII. 266–267 (1897)

On a Method of Studying Displacement.

By R. F. MUIRHEAD, MA., B.Sc.

Let a rigid body F suffer a displacement to a new position, where it will be denoted by F'

Then any point A belonging to F will take up a position A′ in F'. Let B be the point of F which coincides with A′. Then corresponding to B in F there will be a point B′ in F'. Let this again coincide with C in F, and so on. We have thus a sequence or chain of points whose relation may be thus indicated

$$\begin{array}{l} A\,B\,C\,D\,E\,F\,G\ \ldots\ldots \\ \quad A'B'C'D'E'F'G'\ \ldots\ldots \end{array}$$

That is to say, the figure ABCDEFG . . . in the body F assumes the position A′B′C′D′E′F′G′ . . . in F' and in such a manner that the figure BCDEFG . . . coincides with A′B′C′E′F′ . . .

That the construction may actually exist it would be necessary, in general, that the body F should be indefinitely extended on all sides.

The construction just given forms the basis of the method of studying displacement which I wish to put forward here.

It is at once obvious that AB = BC, for A′B′, which is merely AB displaced, coincides with BC. Similarly BC = CD = DE = etc.

Also the angles ABC, BCD, CDE, etc., are all equal for a similar reason, and the dihedral angles between each successive pair of the planes ABC, BCD, CDE, etc., are all equal. In fact the figure ABCDEF . . . is a sort of regular rectilinear helix, which may degenerate, if the angles between the successive planes are zero, into a regular plane polygon, which will ultimately be closed or not, according as the angle ABC is or is not commensurable with a right angle.

Further, the figure may degenerate, if ABC = 180°, into a series of equidistant collinear points.

The construction just explained applies equally to displacements in spaces of one, two, three, or more dimensions.

It enables us as it were to *produce* a displacement indefinitely, and may therefore be called the method of displacement-production ;

and the resulting figure may be called a displacement-sequence or displacement-chain. Any part of it is congruent with any other part of equal extent. Thus in solid space, the tetrahedron ABCD is congruent with the tetrahedron BCDE, and with the tetrahedron DEFG.

In unidimensional displacement, the sequence AB determines the displacement completely, in two-dimensional motion the sequence ABC does so; in three-dimensional, the sequence ABCD, and so on.

The case of one-dimensional displacement, as of a line-body displaced along itself, is so simple as to require no special study.

Two-dimensional Displacement.

Take the case of motion of a plane body in a plane. Let ABCD be a displacement-chain.

Bisect the angles ABC, BCD by lines BO, CO, meeting in O.

Then $\angle OBC = \frac{1}{2} \angle ABC = \frac{1}{2} \angle BCD = \angle BCO$

$$\therefore \quad BO = CO.$$

But when B is displaced to C, the bisector BO comes to CO. Hence O coincides with itself after displacement. Thus in any plane displacement there is one point that remains unmoved.

Corollary. Every plane displacement is equivalent to a rotation about a fixed point, the angle of rotation being the supplement of $\angle ABC$.

Another Construction. Let M and N be the mid-points of AB and BC, so that MN is a sequence.

Let MO $\perp^r$ to AB and NO $\perp^r$ to BC be drawn, meeting in O. Join OB.

In the triangles OBM, OBN, we have $MB = BN$ and OB common, and $\angle OMB = \angle ONB = 90°$.

Hence $OM = ON$.

But the line MO is displaced into NO. Hence, as before, O remains unmoved, and the fundamental theorem of plane displacement is proved.

As examples of other simple applications of displacement-production in two dimensions, consider the following.

(1) If AB, BC are collinear, the same is true of the sequence of every other point in the body. Let $\alpha\beta\gamma$ be such another sequence.

Join $A\alpha$, $B\beta$. Then the figure $BA\alpha$ displaces into $CB\beta$

$$\therefore \quad \angle BA\alpha = \angle CB\beta.$$

And $A\alpha = B\beta$, since the one displaces into the other.

Thus $\quad A\alpha$ is equal and parallel to $B\beta$

$\therefore \quad \alpha\beta$,, ,, ,, ,, ,, AB

Similarly $\beta\gamma$,, ,, ,, ,, ,, AB

$$\therefore \quad \alpha\beta\gamma \text{ are collinear.}$$

We see from this that in such a displacement, every point has a displacement equal and parallel to that of A.

(2) If any two points A and α move through equal and parallel distances AB and $\alpha\beta$, in the same sense, so do all the others.

Let $\alpha'\beta'$ be any other sequence.

Then $A\alpha\beta B$ is a parallelogram, $\therefore$ $A\alpha$ is $\parallel$ to $B\beta$.

Again $\quad \angle A\alpha\alpha' = \angle B\beta\beta'$.

$\therefore$ $\alpha\alpha'$ and $\beta\beta'$ are parallel; and they are equal:

Hence $\quad \alpha'\beta'$ is $=$ and $\parallel$ to $\alpha\beta$.

(3) If any line remains parallel to itself, so do all lines that can be drawn in the body.

Let $A\alpha$ be parallel to its displaced position $B\beta$: the two are obviously equal. Hence $A\alpha\beta B$ is a parallelogram. Hence as in (2) every other point-displacement is equal and parallel to AB and $\alpha\beta$.

Three-dimensional Displacement.

To establish by this method the fundamental theorem of displacement in three dimensions, the following Lemma in Solid Geometry is required, of which an elementary proof is given later.

Lemma: Let BCYX be a tetrahedron such that

$$\angle XBC = \angle BCY \text{ and } \angle CYX = \angle YXB,$$

then it follows that $CY = BX$, also $CX = BY$, also the dihedral angles at equal edges are equal.

Proposition: In every displacement of a body in solid space, there is *one* straight line of the body which is displaced along itself.

Let ABCDE be a displacement-chain, and let BX, CY, DZ be

the interior bisectors of the plane angles ABC, BCD, CDE; and let XY be the shortest distance between BX and CY, *i.e.*, the line which is perpendicular to each of them.

We have $\angle \mathrm{XBC} = \frac{1}{2} \angle \mathrm{ABC} = \frac{1}{2} \angle \mathrm{BCD} = \angle \mathrm{BCY}$

Hence by the Lemma, $\mathrm{BX} = \mathrm{CY}$.

Hence XY is a sequence; and the next point in this sequence is on DZ: let Z be that point; then YZ is the shortest distance between CY and DZ.

We shall show that X, Y, Z are collinear. Since the angles CYZ and CYX are right, it only remains to show that they are in the same plane.

Now by the Lemma the dihedral angles at the edges XB and YC are equal, in the tetrahedron XBCY; and the dihedral angle at CY in the tetrahedron YCDZ being congruent with the former, is equal to the latter.

Thus the planes XYC and ZYC make equal angles with the plane BCDY, and are on opposite sides of it. Hence they are parts of the same plane.

Hence XY and YZ are collinear.

Thus in the given displacement the line XYZ is displaced along itself.

Corollary. The displacement is equivalent to a screw motion, the pitch of the screw having the same ratio to XY that 360° has to the angle between the planes BXY and XYC.

Another proof, perhaps more intuitive but less complete, is as follows:

If we draw, from some fixed point, lines in the same direction as AB, BC, CD, etc., respectively, the angles between those lines, taken successively in pairs, are all equal, and also the angles between the planes containing successive pairs. Thus we get a figure which bears the same resemblance to a circular cone that a regular (closed or unclosed) polygon bears to a circle. Obviously a line equally inclined to three of these lines is equally inclined to all.

Now the shortest distance XY of the previous figure is clearly equally inclined to AB and BC, since it is perpendicular to the bisector BX. Similarly it is equally inclined to BC and CD. Thus

it is equally inclined to three links of the chain, and, therefore, to all. Hence it is in the same direction as the shortest distance between CY and DZ.

Again, the two successive shortest distances are coterminous in Y, since by symmetry $BX = CY$. Other simple applications to three dimensions are as follows:—

(1) If ABC are collinear, then either ABC is the axis of the screw, or else the displacement is a pure translation.

(2) If ABC are collinear and $\alpha\beta \equiv AB$, then there is translation only (using the symbol $\equiv$ to denote that the two lines are $=$, $\parallel$ and in the same sense).

For if we take any third sequence $\alpha'\beta'$, the tetrahedra $\alpha'AB\alpha$ and $\beta'BC\beta$ being in sequence, are congruent; and their bases αAB and βBC being in the same plane, we have clearly $\alpha'\beta' \equiv AB \equiv \alpha\beta$.

(3) If three sequences, AB, A′B′, $\alpha\beta$, are such that $AB \equiv A'B' \equiv \alpha\beta$, then all sequences of two points are so.

For if $\alpha'\beta'$ be any other sequent pair, the tetrahedra $AA'\alpha\alpha'$ and $BB'\beta\beta'$ being congruent and on homothetic bases $AA'\alpha$ and $BB'\beta$, $\therefore\ \alpha'\beta' \equiv \alpha\beta \equiv$ etc.

This fails only when $AA'\alpha$ are collinear, in which case the second datum is not independent of the first, so that the case reduces to (4).

(4) If there are two sequences AB and $\alpha\beta$ such that $AB \equiv \alpha\beta$, then the axis of the screw must be parallel to $A\alpha$, or else the displacement is a pure translation.

For if we take any other line in the body $\equiv A\alpha$, say $A'\alpha'$, then $A\alpha\alpha'A'$ and $B\beta\beta'B'$ are parallelograms, as also $A\alpha\beta B$. Hence

$$A'\alpha' \equiv A\alpha \equiv B\beta \equiv B'\beta'.$$

Thus any line $\parallel$ to $A\alpha$ remains so. Hence $A\alpha$ is parallel to the screw-axis, if there is one.

(5) If ABCD are coplanar but not collinear, two cases may occur (i.) the plane ABC of the body may be facing the same way in its displaced position BCD, or (ii.) it may have been turned over; *i.e.*, the angle between successive planes in sequence may be 0° or 180°. In the former case, all points in the plane ABC evidently remain in it, and therefore (as in uniplanar displacements) one

point in the plane is unmoved, and the displacement is a rotation about an axis through this point perpendicular to the plane ABC.

In the latter case, the sequence ABCD . . . forms a zig-zag figure, consisting of equal distances which are parallel alternately, so that AB, CD, EF, etc., are parallel to one another, and also BC, DE, etc., are parallel to one another. It follows that ACEG . . . are collinear and parallel to BDFH . . . . This indicates that the axis of the screw is the parallel line midway between ACE and BDF, and that the rotation about this axis is 180°.

If we restrict our attention to the plane body ABCD in this case, we see that it corresponds to *inversion* in a plane, and we can at once deduce several useful propositions with respect to inversion of plane figures: *e.g.*, that when a plane figure is inverted, there is *one* direction which remains unaltered, and another at right angles to it which is exactly reversed: also that one line in the former direction slides along itself, but all those in the latter direction are displaced by the same amount parallel to themselves.

It may be remarked that in general an inversion in any space is equivalent to a special case of displacement in the next higher space.

Let us now apply to three-dimensional displacement a construction similar to the second construction in two-dimensional displacement. Let P, Q, R, S be the axial planes of the pairs of points AB, BC, CD, DE respectively, ABCDE being a sequence, and the axial plane being the locus of points equidistant from each point of the pair in question.

P, Q, and R intersect in a point O. Then O is equidistant from A, B, C, D.

Similarly Q, R, S intersect in a point O′ equidistant from B, C, D, E.

Now compare the two tetahedra OBCD and O′BCD. Obviously their corresponding edges are equal, each to each. Are they then congruent? If so it would follow that O and O′ were coincident, and therefore that in three-dimensional displacement there is a point that remains unmoved, which is not so.

The explanation is that OBCD and O′BCD are *symmetric*, not *congruent*, *i.e.*, they bear to one another the same relationship that a body bears to its image in a plane mirror. We may say that one is the *reflex*, or *reverse* of the other. To prove this we may note

that O and O′ are the centres of the circumspheres of ABCD and of BCDE respectively. If we can show that A and E are on opposite sides of the plane BCD, it will follow that O and O′ are also on opposite sides of that plane.

We note that the tetrahedron EBCD is congruent with DABC, the corresponding points being in the order stated. Now if the circuit ABC, when looked at from D, is clockwise, we see that the circuit BCD, looked at from A, will be counter-clockwise, and *vice versa*. But EBCD is congruent with DABC, and therefore the circuit BCD, seen from E, is in the same sense as the circuit ABC, seen from D, therefore in the sense opposite to that of BCD seen from A. Hence A and E must be on *opposite* sides of the plane BCD. Thus the explanation is complete.

Noting that in one-dimensional displacement, and in three-dimensional displacement there is a line which coincides with itself, but no self-coincident point, while in two-dimensional motion, the reverse is the case, we might suspect that for displacement in space of *even* dimensions there is always one point fixed, but no self-coincident line, while the reverse is the case for displacement in space of *odd* dimensions.

This is true, and it can further be shown that in n-dimensional displacement that is always a flat $\overline{(n-2)}$-fold space which moves in itself, from which it follows that there is an $\overline{n-4}$-fold flat space in *that* which moves in itself, and so on, down to a *point* in the case of n even, or a *line* in the case of n odd.

From this we deduce the remarkable result that the most general displacement in $2n$-fold space is equivalent to n rotations, and that in $(2n+1)$-fold space the most general displacement is equivalent to n rotations together with a translation.

For, calling a flat n-fold space S^n for brevity, we may reason thus:—in the case of displacement in S^{2n} we begin by keeping fixed the S^{2n-2} which is perpendicular to the self-coincident plane of the body and meets it in the self-coincident point. A rotation about this S^{2n-2} will bring all points of the said plane into their final positions. Next keep fixed the S^{2n-2} which is perpendicular to the self-coincident S^4 and meets it in the plane before mentioned. This gives rise to a rotation which brings every point of the S^4 into its final position. Proceeding thus we finally reach the displaced position of the body by means of n rotations.

In the case of displacement in a S^{2n+1} we may begin by a displacement along the self-coincident line, bringing its points into their final positions.

But the proof, or even the complete statement of these matters would unduly weight this paper, which aims rather at explaining the method than at exhausting its applications; so for the present I shall content myself by giving the elementary proof of the statement that in a space of even dimensions there is one point that does not move.

It is obvious that an extension of the second construction given in two-dimensional displacement goes far towards this proof. We must in n-dimensional displacement take a chain of $n+2$ points A, B, C . . . H, K, L, and construct the axial S^{n-1} for each successive pair. Let these axial S^{n-1}s be P, Q, R . . . T, U. They are $n+1$ in number. The first n of them intersect in a point O, and the last n of them in a point O′. As in the case of three dimensions, O and O′ will be coincident or not according as A and L lie on the same or on opposite sides of the S^{n-1} determined by BC . . . HK. This again will depend on whether the figure BC . . . HK has the same sense as seen from L and from A or not. But since KABC . . . H and LBC . . . HK are congruent, ABC . . . H seen from K and BC . . . HK seen from L have the same sense. Thus O and O′ will be coincident or not according as BC . . . HK seen from A and ABC . . . H seen from K have the same sense or not, or finally according as the figure ABC . . . HK, which is a *simplissimum* in n dimensions, has not or has its sense altered by changing the order of its points to KABC . . . H.

This implies a conception for n dimensions equivalent to that of the distinction between a positive and a negative circuit in a plane. If we form such a conception, we find that the interchange of two adjacent letters changes the *sense* of the figure. Hence an *odd* number of such interchanges will result in a figure of *opposite sense*, and an *even* number in one of the *same sense*. Now in the "circuit" ABC . . . HK there are $n+1$ letters, so that n interchanges are required to make the change to KABC . . . H. Hence the sense of the figure is altered or not according as n is odd or even.

Hence if n be even O and O′ coincide, that is there is *one* point unmoved in the general displacement.

I may mention that in the *Annales de l'École Polytechnique de*

Delft, t. VII. (1891), pp. 139–158, Mr P. H. Schoute has a paper entitled "Le Déplacement le plus général dans l'espace à *n* Dimensions," in which he demonstrates the resolution above stated in connection with the general theory of projectivity in S^n by the aid of homogeneous coordinates. In this paper, which is a very interesting and elegant one, he refers to an earlier paper on the same subject by Rahusen, t. IV., p. 104, of the same *Annales*, and also to the treatise by G. Veronese entitled "Fondamenti di Geometria."

Appendix: Elementary Geometry of the Isosceles Skew Trapezium.

Definition: Let a skew quadrilateral ABCD having $\angle A = \angle B$ and $\angle C = \angle D$ be called an Isosceles Skew Trapezium.

Prop. 1. In such an isosceles skew trapezium $AC = BD$ and $AD = BC$.

Construction: Draw $AE \equiv DC$, also $BF \equiv CD$ (using the symbol $\equiv$ to mean equal and in the same direction).

Join CE, EB, AF, FD. Then AECDFB is a triangular prism.

Now the trihedral solid angle A (BDE) is congruent with the solid angle B (ACF), for

$$\angle EAD = 180° - \angle ADC = 180° - \angle DCB = \angle CBF$$
$$\angle EAB = \angle FBA$$
$$\angle DAB = \angle CBA$$

whence the congruity follows by Bk. XI. Additional Prop. 1 of the *Pitt Press Euclid.*

$\therefore$ dihedral angle $E\overline{AB}D$ ($\overline{AB}$ being the edge) = dihedral angle $F\overline{BA}C$

$\therefore$ dihedral $D\overline{AB}F$ = dihedral $C\overline{AB}E$

Again the trihedrals A(DBF) and B(CAE) have

$$\left.\begin{array}{l} \angle BAE = \angle ABF \\ \angle DAB = \angle CBA \\ \text{dihedral } D\overline{AB}F = \text{dihedral } C\overline{AB}E \end{array}\right\}$$

Hence by Add. Prop. 2, Bk. XI. of the *Pitt Press Euclid* they are congruent.

$\therefore \angle DAF = \angle CBE.$
$= \angle DFA$

$\therefore DA = DF = CB = CE.$

But the Δs DAB, CBA are congruent by Euc. I., 4. $\therefore CA = DB.$

Prop. 2. Converse of Prop. I. If $\angle C = \angle D$ and $CB = DA$, then ABCD is an isosceles skew trapezium.

For the Δs DCB, CDA are congruent by Euc. I., 4.

$\therefore$ „ „ ABC, BAD „ „ „ „ I., 8.

Q. E. D.

Prop. 3. Another converse of Prop. I. If $CB = AD$ and $CA = BD$, the same result follows easily by Euc. I., 8.

Prop. 4. In an isosceles skew trapezium ABCD as above, if H, K be the mid points of CD, AB, then HK is perpendicular both to AB and to CD.

The triangles BCH, ADH are congruent, by Euc. I., 4.

$\therefore$ $HA = HB$. $\therefore$ $\angle HKA = \angle HKB$.

$\therefore$ HK is $\perp^r$ to AB.

Similarly HK is $\perp^r$ to CD.

Thus HK is the shortest distance between AB and CD.

Prop. 5. If XY be the shortest distance from AD to CB, then A and B are equally distant from X and from Y respectively; and so also are C and D.

This is a corollary of Prop. I.

Conversely, an isosceles skew trapezium may be constructed in two ways, thus :—

(1) Take two pairs of points (A,B), (D,C) on two lines, BCX, ADY in space, whose shortest distance is XY, such that $CY = DX$ and $BY = AX$. Then ABCD is an isosceles skew trapezium having $\angle C = \angle D$ and $\angle A = \angle B$.

(2) Take two lines AB, CD in space and let HK be their shortest distance, C and D being equally distant from H, A and B being equally distant from K. Then ABCD is an isosceles skew trapezium having $\angle A = \angle B$ and $\angle C = \angle D$.

Figure (34) shows the isosceles skew trapezium ABCD in relation to its shortest distances XY and HK.

It is obvious that HK is the locus of the mid points of lines joining points on XA, YB, which are equally distant from X and Y respectively: also that the figure is symmetrical about the line LHK.

On Superposition by the Aid of Dissection.

By R. F. Muirhead, M.A., B.Sc.

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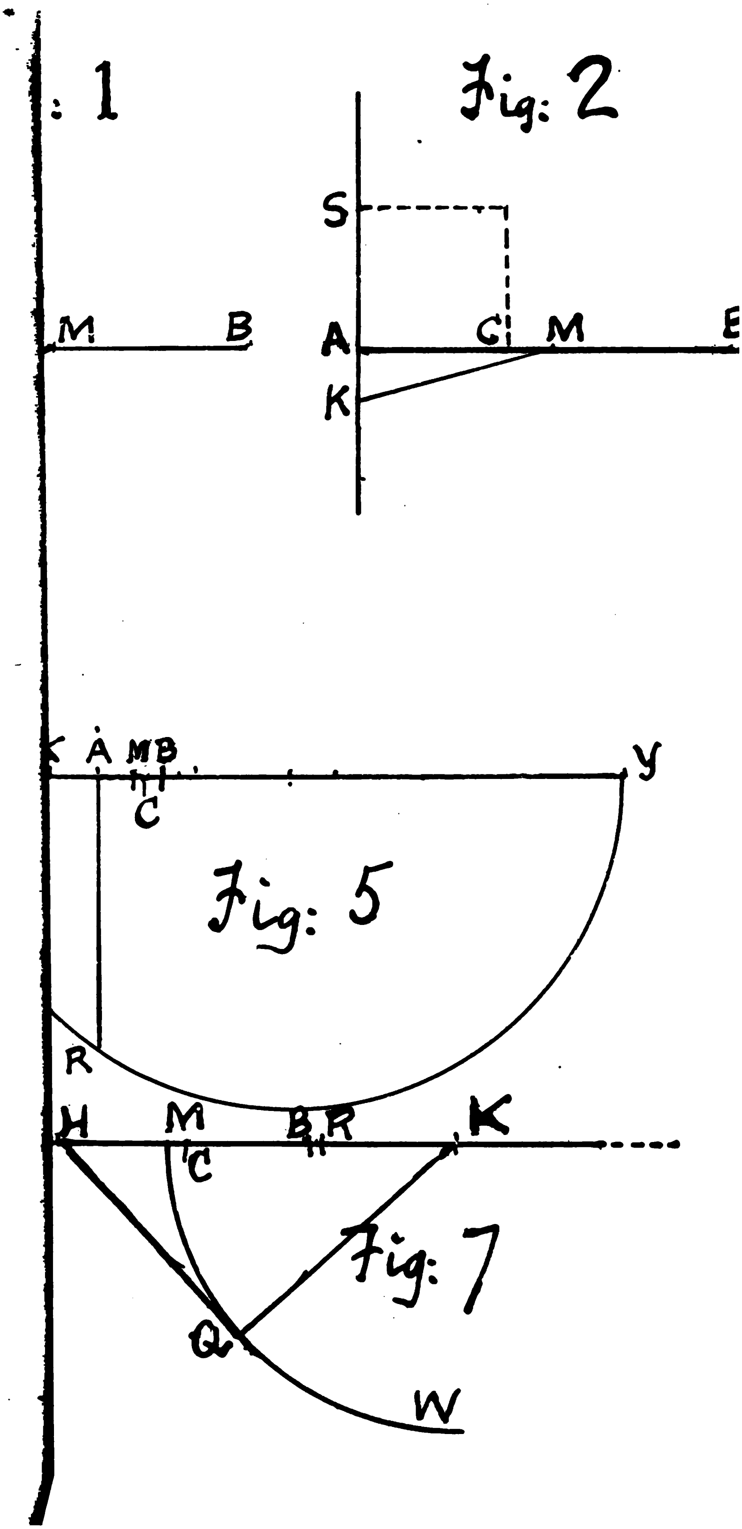

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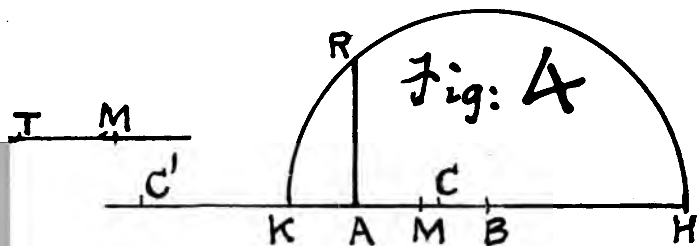

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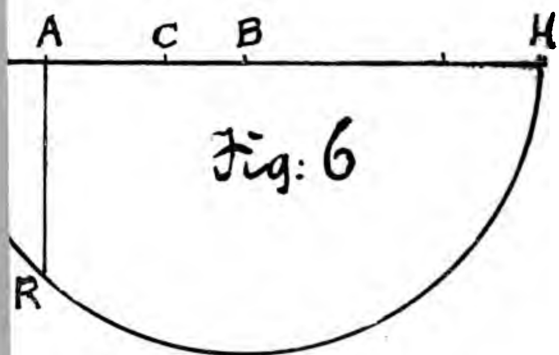

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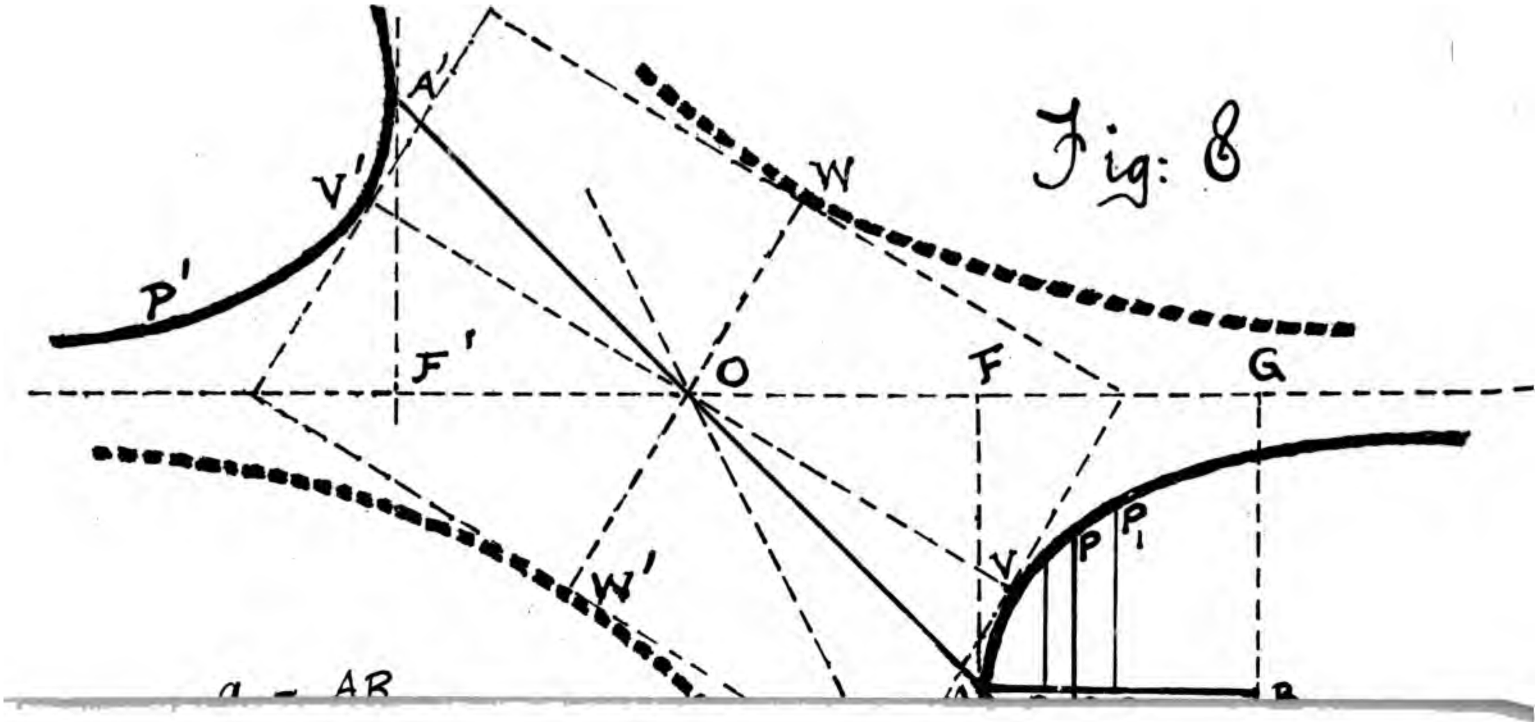

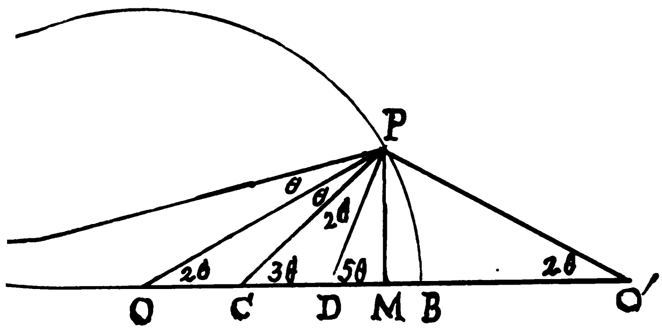

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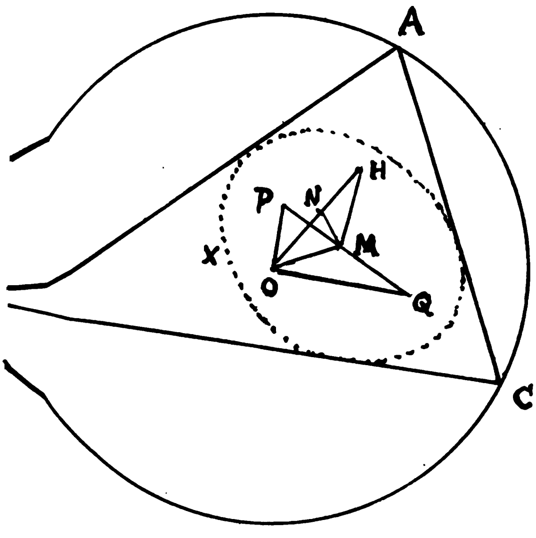

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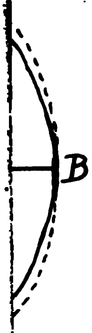

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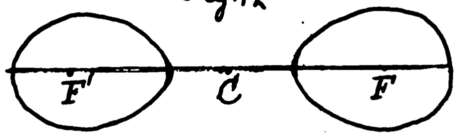

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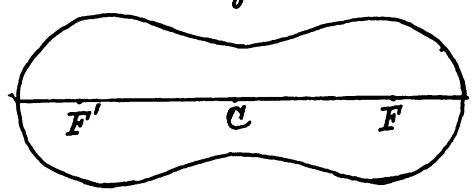

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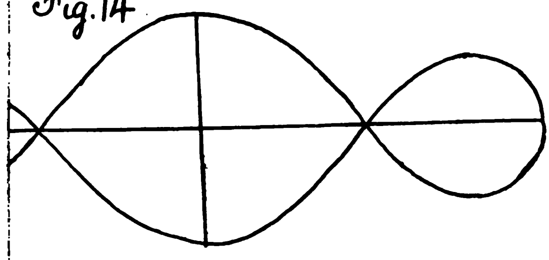

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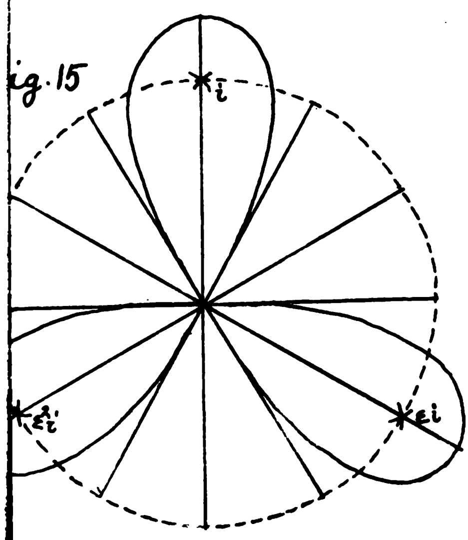

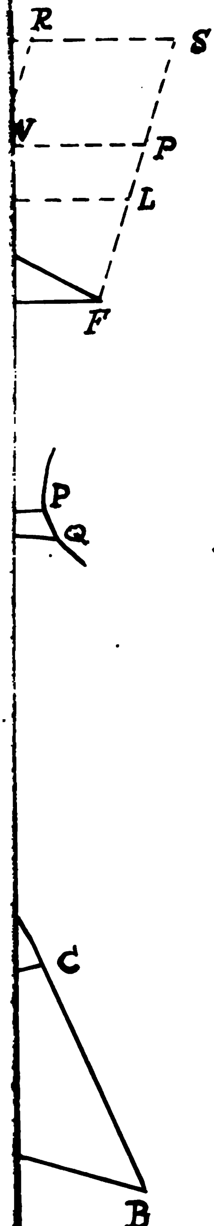

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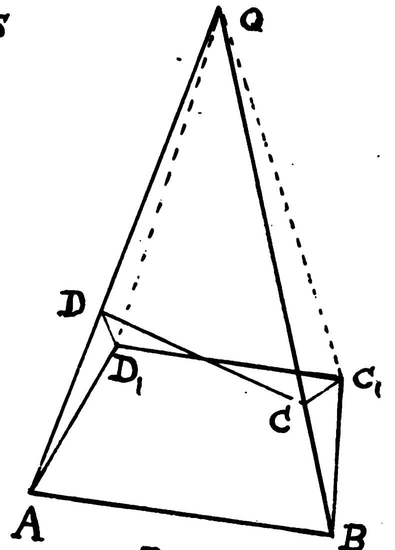

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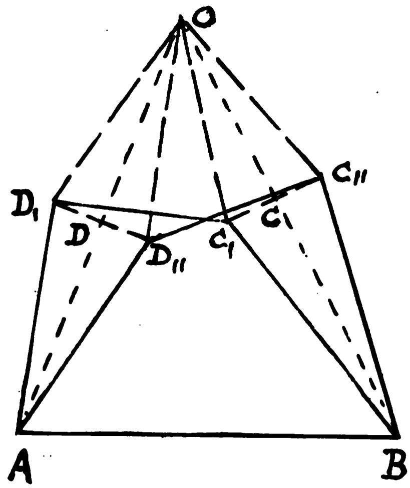

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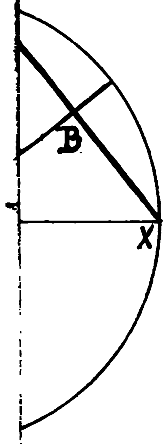

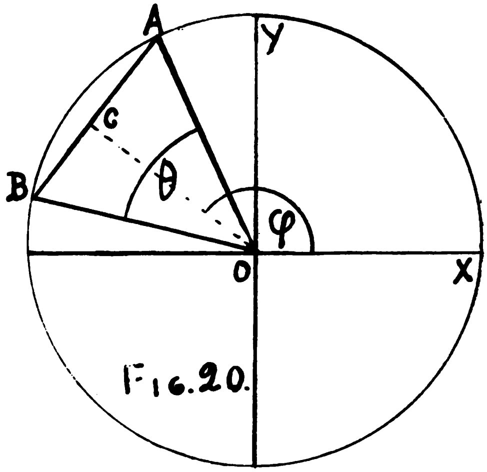

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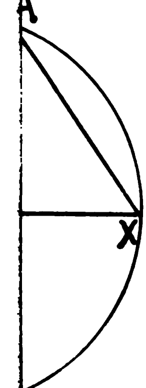

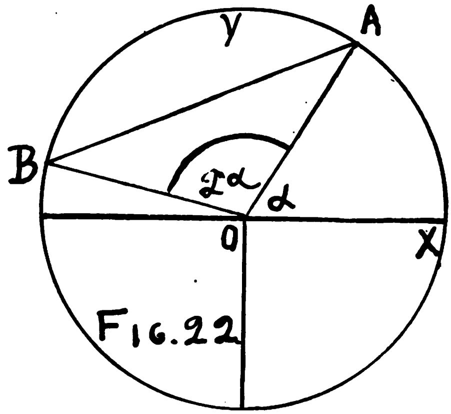

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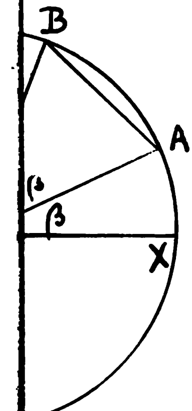

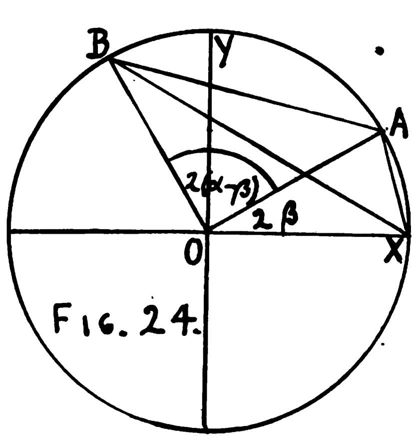

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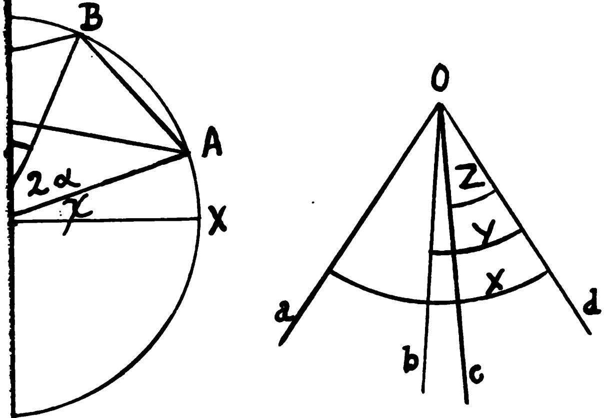

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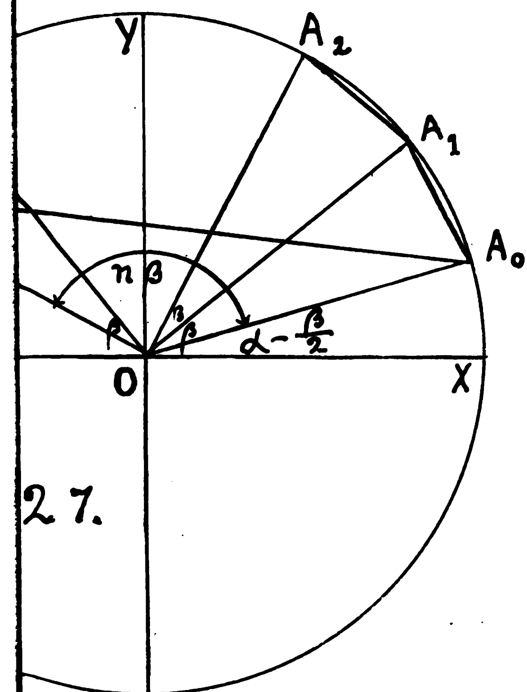

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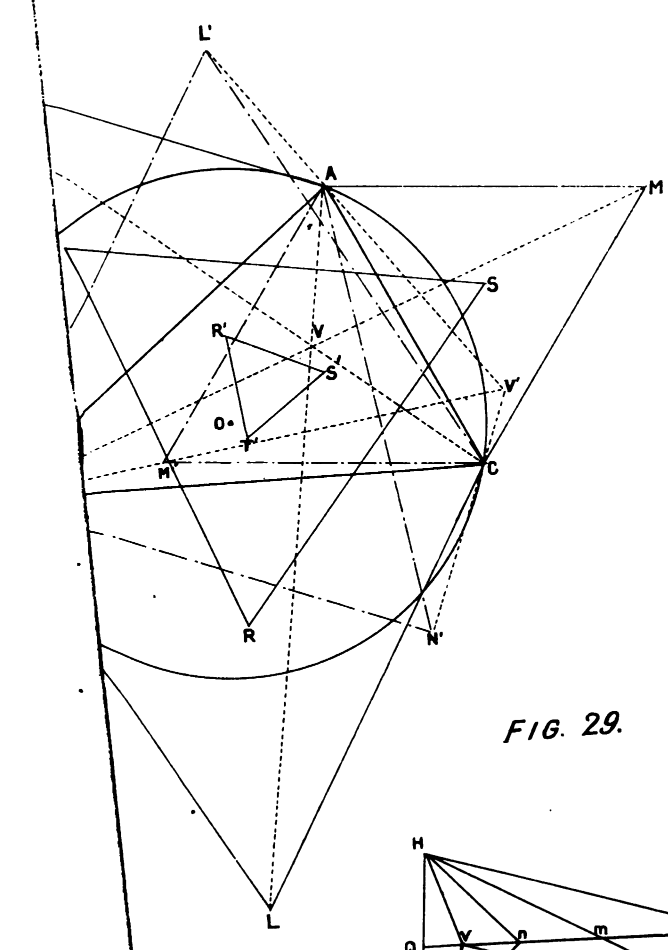
FIG. 28.



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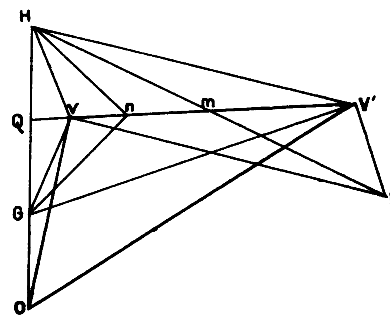

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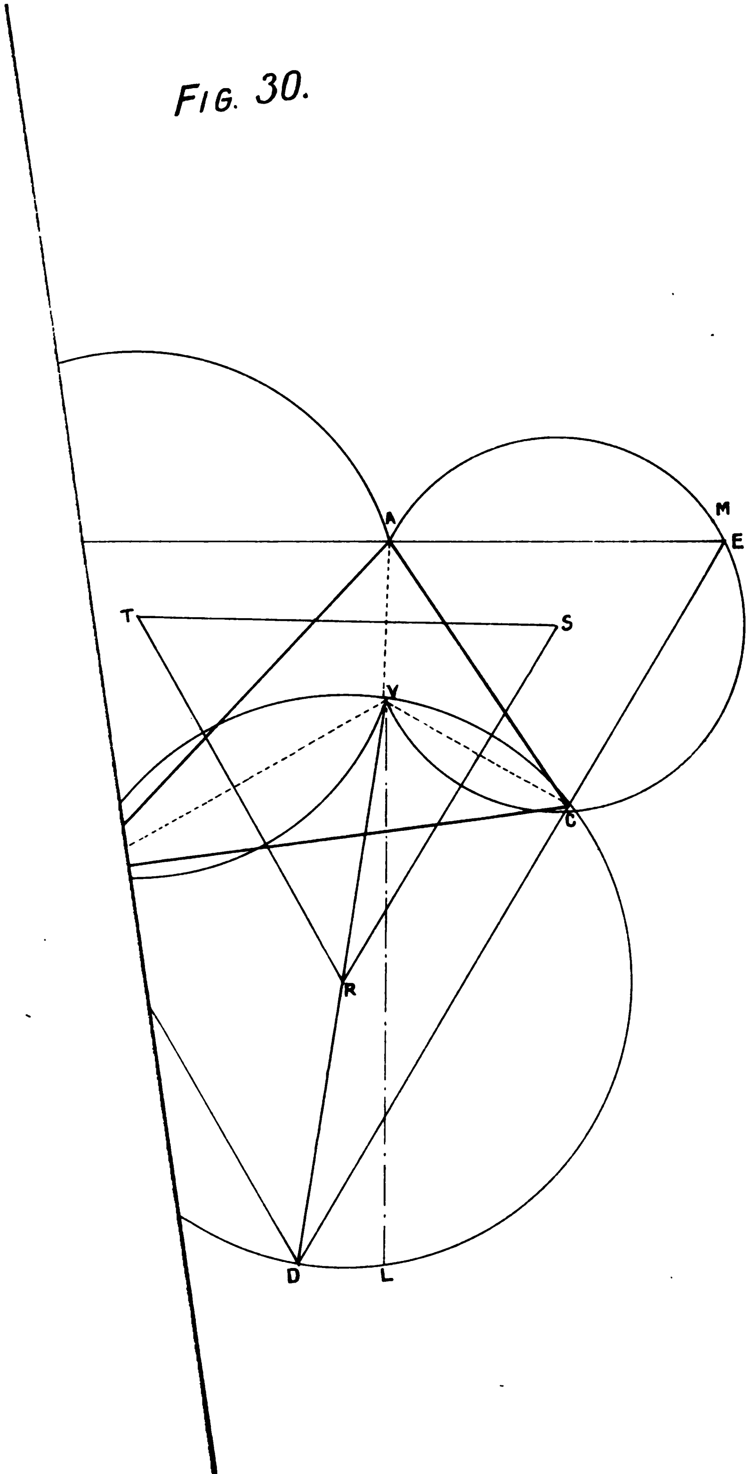

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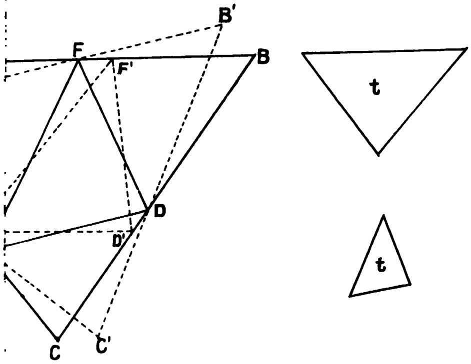

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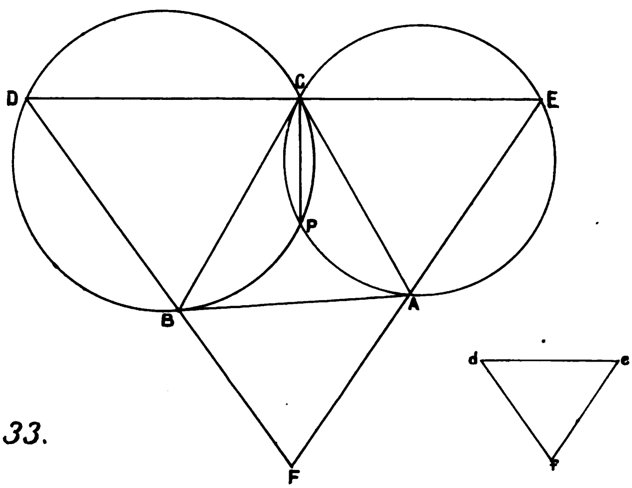

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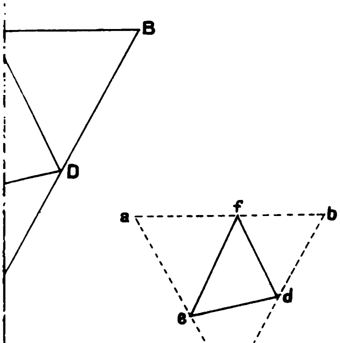

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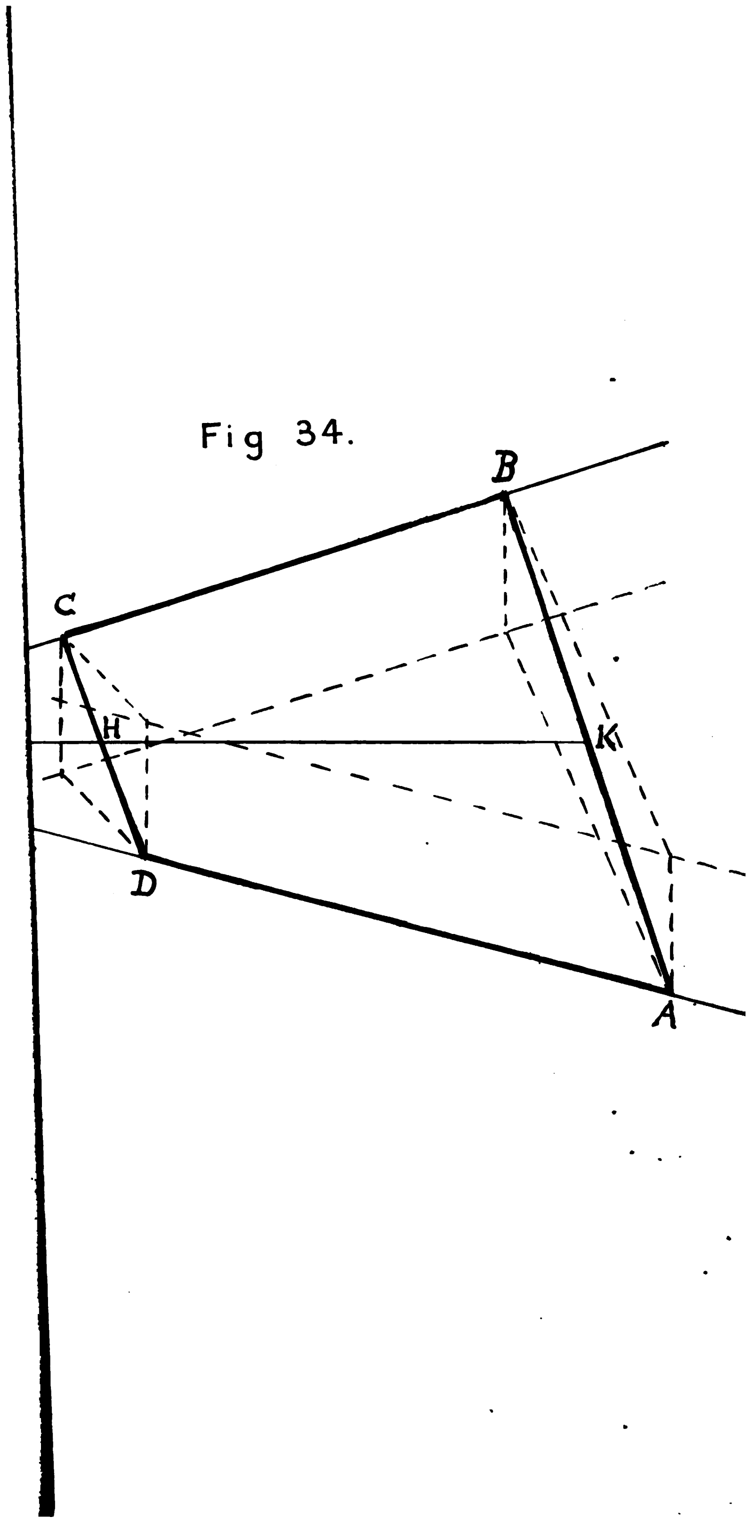

PROCEEDINGS

OF THE

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VOLUME XVI.

SESSION 1897-98.

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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

SIXTEENTH SESSION, 1897-98.

First Meeting, 12th November 1897.

J. B. CLARK, Esq., M.A., F.R.S.E., Vice-President, in the Chair.

For this Session the following Office-bearers were elected :—

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The Treatment of Arithmetic Progressions by Archimedes.

By Professor Gibson.

The following paper was written last summer, and was submitted to Dr Mackay with a view to eliciting his opinion particularly on the curious passage referred to in § 3, and on the remarks contained in § 8. I was not aware of the intention of Mr T. L. Heath to follow up his excellent edition of Apollonius by an edition of Archimedes on similar lines, and when I saw the announcement of his Archimedes in the month of October, I at once concluded that the notes I had made would have been anticipated by him. Since reading his masterly work, however, I am disposed to think there is still sufficient interest in the notes I have written to justify me in laying them before the Society; I therefore submit them in their original form, although I should have omitted certain details had I been acquainted with Mr Heath's work before writing the paper.

1. In his books *On Helices* and *On Conoids and Spheroids*, Archimedes has effected the evaluation of areas and of volumes by methods which are very closely analogous to the modern algebraical methods, depending as they do largely on the sum of arithmetical progressions. The main difference is to be found in the almost exclusive use by Archimedes of inequality theorems which are required for the application of the method of exhaustion, while the modern treatment replaces this by a more or less rigorous use of convergent series. This peculiarity has to a certain extent obscured the really complete command he had of such progressions, but a careful study of his works is sufficient to show that, where he did not enunciate his theorems in forms giving the sum of the series in closed terms, he took this course, not from inability to give the closed form, but chiefly because the inequality theorems were those of which almost exclusively he was in need, and also because their statement was much more concise. The undoubted prolixity of enunciation and demonstration is due much more to the want of an algebraic symbolism than to anything in the method of proof; and

one can hardly be surprised that he should have chosen the simpler form for his theorems, when nothing was to be gained but rather something would have been lost by enunciating them as equalities.

2. In presenting the methods of Archimedes it is, I think, a mistake to adopt the modern method of using symbols merely for the first term, the common difference, and the number of terms, as the real simplicity of the proofs is thereby obscured. The chief defect of his notation—and it is one that causes great prolixity both in statement and in demonstration—is the absence of a symbol for the number of terms. In this paper, therefore, I use distinct letters to represent the terms of the series; in Archimedean language the terms are straight lines, and these are specified sometimes by one letter only (as in this paper) and sometimes by two; I employ, however, a symbol for the number of terms as well as the modern algebraic or geometric symbolism. Throughout the paper the first term of the arithmetic progression will be denoted by a, the second by b, etc., the last by l, the second last by k, etc., while, unless it be otherwise stated, the number of terms will be n; it will also be supposed, unless otherwise specified, that a is the greatest and l the least term.

3. Though Archimedes enunciates (*Opera* I., p. 290*) and repeatedly uses the theorem that when l is the common difference twice the sum of the n lines $a, b, c, \ldots l$ is greater than na and twice the sum of the $n-1$ lines $b, c, \ldots l$ is less than na, he gives no distinct proposition in proof of these inequalities. But in the 11th Proposition of the Book *On Helices* he indicates how they may be proved, and in the course of the 10th Proposition of the same book he states and proves the exact theorem, namely—

$$2(a+b+\ldots+l)=n(a+l);$$

further, in this same proposition he uses this value for the sum repeatedly, so that he was evidently quite familiar with it.

On the other hand, it is rather curious that in the only proposition where he requires to use the exact value, he has, if we accept the text of Heiberg, fallen into error. In *Conoids and Spheroids*, Prop. 21 (*Opera*, I., p. 392) he has to compare the sum of the

* The references are to Heiberg's edition.

$n-1$ lines $b, c, \ldots l$ with $(n-1)a$, and he says that $(n-1)a$ is greater than twice the sum of $b, c, \ldots l$, though his own diagram shows clearly the equality of the two expressions. The two sentences in lines 14–18 are quite conclusive as to the mistake, and the editor, by referring to page 290, where the inequalities are stated, leads one to infer that he, too, has failed to notice the slip. Archimedes would seem not to have observed that the number of lines $b, c, \ldots l$, is the same as the number of lines a with which he was comparing them. The slip is no doubt a trivial one, and does not affect the final conclusion, but it appears to indicate the subordinate position which the exact theorem occupied in his collection of results. *

It will be noticed that the least term is assumed to be equal to the common difference; it will be seen later how Archimedes gets over that restriction when a series occurs not satisfying that condition.

4. The most important theorems are those dealing with the sums of squares, and it was possibly the summation of the series that constituted a portion of the difficulties referred to in the letters to Dositheus, prefixed to the books on *Helices* and *Conoids and Spheroids.*

The use to which these series are put may help to explain their origin. In finding the volume of a segment of a conoid or spheroid, Archimedes employed three sets of cylinders.

---

* The text of Heiberg in this passage differs considerably from that of Torelli, but it is hardly possible that the latter can be correct. The sentence (*Torelli*, p. 287, at foot) "Ἆρα καὶ ὁ ὅλος κύλινδρος κ.τ.λ" is a mere repetition of that preceding it, while the position of ἄρα at the beginning of the sentence is at variance with Greek usage. The deletion of πολλῷ before ἄρα is stated in Heiberg's note to be due to Commandine, and it is easy to understand the deletion, for in Torelli's text the inscribed figure is only compared with the *whole* of the circumscribed cylinder and not also with a part of it as in Heiberg's text. In all the texts there is a certain ambiguity as to the precise meaning of the phrases "all the lines" and "all the lines cut off between AB, BΔ, but lines 14–16 in Heiberg make the meaning quite clear, for there it is explicitly stated that the circumscribed cylinder diminished by one of its elementary cylinders is more than double of the inscribed figure while it is obviously exactly double. Heath's rendering of the proposition is, of course, quite accurate in its mathematics, but in the condensation of the original text the erroneous statement has apparently been overlooked. So far as I am aware, the slip has not been previously pointed out.

The first set consisted of a single cylinder, K, whose axis was that of the segment, whose lower base was the base of the segment and whose upper base was in the tangent plane parallel to the base of the segment.

The second set, C, formed a figure circumscribing the segment; it was built up of cylinders of equal altitudes, with generators parallel to the axis of the segment, whose lower bases were the base of the segment and the sections in which the segment was cut by planes drawn parallel to its base through points dividing its axis into any number, say n, of equal parts.

The third set, I, formed a figure inscribed in the segment in the same way as the circumscribed figure C.

C, containing n cylinders, is greater than the segment, and I, containing $n-1$, is less. Archimedes shows that $\mathrm{C}-\mathrm{I}$ can be made less than any given solid, and he finds limits for the ratios of C to K and of I to K. In finding these limits he has to sum the series

$$a^2+b^2+ \quad . \quad . \quad +l^2.$$

It is perhaps worth remarking that if the method just described be applied to the known theorem (*Euclid* XII., 7) that a pyramid on a triangular base is a third of the prism on the same base and of the same altitude, the inequality theorems may be at once deduced. The pyramid and prism being supposed to have a common edge divided into n equal parts, and planes being drawn through the points of section parallel to the base, the sets C, I may be taken as prisms with edges parallel to the common edge, while the whole prism will represent K. If a be one side of the triangular base, the sides of the triangular sections parallel to a will with a form an A, P, $a, b, \ldots l$, and we shall have

$$\frac{\mathrm{C}}{\mathrm{K}}=\frac{a^2+b^2+\ldots+l^2}{na^2}, \quad \frac{\mathrm{I}}{\mathrm{K}}=\frac{b^2+\ldots+l^2}{na^2}.$$

But $$\mathrm{C}/\mathrm{K}>\tfrac{1}{3} \text{ and } \mathrm{I}/\mathrm{K}<\tfrac{1}{3};$$

hence $$3(a^2+b^2+\ldots+l^2).>na^2>3(b^2+\ldots+l^2).$$

Whether this theorem on the relation between the pyramid and prism which Archimedes himself cites (*Opera* I., p. 4) as one established by Eudoxus in a manner generally accepted as sound, may have led him to these inequalities, can only be matter of conjecture. In any case, his proof is quite different, as will now be shown.

5. The 10th Proposition of the Book *On Helices* is as follows, l being the common difference:—

$$3(a^2+b^2+\dots+l^2)=(n+1)a^2+l(a+b+\dots+l)$$

and the corollary is $3(a^2+b^2+\dots+l^2)>na^2>3(b^2+\dots+l^2)$

The proof is very peculiar. It is obvious that

$$a=b+l=c+k=d+j=\text{ etc. }=l+b$$

Hence squaring the $n-1$ values of a, adding results and increasing each side of the equation by $2a^2$ he gets

$$(n+1)a^2=2(a^2+b^2+\dots+l^2)+2(bl+ck+\dots+kc+lb).$$

He next shows that

$$a^2+b^2+\dots+l^2=l(a+b+\dots+l)+2(bl+ck+\dots+kc+lb)$$

by proving that each side of the equation is equal to

$$l(a+3b+5c+\dots+(2n-1)l).$$

Now this transformation is certainly very artificial, but it seems to me not impossible that this last step was really the first in order of discovery.

It may be assumed (Cantor, *Gesch der Math.*, I., p. 153) that Archimedes was familiar with the process of building up a square of side a by starting with a square of side l and adding successively the gnomons $(k^2-l^2),\ (j^2-k^2)\dots(a^2-b^2)$,

and hence that

$$\begin{aligned} a^2 &= (a^2-b^2)+(b^2-c^2)+\dots+(k^2-l^2)+l^2 \\ &= l[(a+b)+(b+c)+\quad+(k+l)+l\ ] \\ &= l[\ a+2(b+c+\quad\dots\quad+k+l)] \end{aligned}$$

since $l=a-b=b-c=\dots=k-l.$

Indeed this is the form in which he expresses the value of a^2 in the transformation, though his proof of this value is no doubt quite different. If the supposition be made that he started from this value of a^2, and the corresponding values for the other squares, he would get for the sum of the n squares

$$\begin{array}{l} l[a+2(b+c+d+\dots+k+l)] \\ +\,l[b+2(\quad c+d+\quad+k+l)] \\ +\,l[c+2(\quad\quad d+\quad+k+l)] \\ +\quad\dots\quad\dots\quad\dots \\ +\,l[k+2(\quad\quad\quad\quad l)] \\ +\,l[l\quad\quad\quad\quad\quad] \end{array}$$

that is,

$$l(a+b+\ldots+l)+2l(\,b+2\,c+3\,d+\ldots+(n-2)\,k+(n-1)l\,)$$

or $\quad l(a+b+\quad +l)+2\,(lb+2lc+3ld+\ldots+(n-2)lk+(n-1)ll)$

or since $\quad 2l=k, \quad 3l=j\ldots(n-2)l=c, \quad (n-1)l=b$

$$l(a+b+\quad +l)+2(lb+kc+jd+\ldots+ck+bl)$$

If this were the form first found for the sum of the squares, the property that each term in the product consisted of two factors whose sum was a would lead to the squaring of the $n-1$ values $b+l$, $c+k$, etc.

The actual demonstration given by Archimedes is no doubt quite different, but the artificiality of the transformation referred to above leads to the suspicion that the traditional method of representing a square as a sum of gnomons may have played a more important part than the completed proof suggests.

The fact that in the enunciation the series

$$a+b+\ldots+l$$

is not summed can not be due to ignorance of that sum, seeing that in the course of the demonstration the summation is repeatedly effected; the reason for the form given seems to be simply that he had no need for the exact sum of the squares in any part of his work, as the inequalities of the corollary contained all he required. Besides, the series are only auxiliary to the determination of areas and volumes; it should not therefore surprise us that he does not put the expression for the sum into a form which his whole discussion shows he might have done had he been treating the series for their own sake.

At the same time, it is to be observed that his substitution of the inequality theorems of the corollary for the exact theorem of the proposition obliges him to treat the ellipsoid in a different way from the hyperboloid, as will be seen in § 8.

6. The theorem of the preceding paragraph assumes the common difference to be equal to the least term, but obviously cases arise where that condition is not satisfied, and Archimedes provides for such cases in the 11th proposition of the same book (*Opera* II., 42–50). The diagram to that proposition makes the common difference equal to the least term, but the enunciation omits the characteristic phrase expressive of this condition and the

demonstration is also independent of it, while the repeated applications of the theorem in the Book *On Helices* show that he understood it in its most general form.

It will be convenient to take the number of terms as $n+1$, and the proposition may then be put in the form

$$\frac{na^2}{a^2+b^2+\ldots+k^2} < \frac{a^2}{al+\frac{1}{3}(a-l)^2} < \frac{na^2}{b^2+\ldots+k^2+l^2}.$$

The theorem will be proved, he says, if it be proved that

$$a^2+b^2+\ldots+k^2 > nal+\tfrac{1}{3}n(a-l)^2 > b^2+\ldots+k^2+l^2.$$

To effect the proof, he subtracts l from each term and thus reduces

the progression $\quad a, b, \ldots k, l \quad [n+1 \text{ terms}]$

to the progression $\quad a-l, b-l \ldots k-l \quad [n \text{ terms}]$

in which the least term $k-l$ is equal to the common difference, and to which therefore the results of Prop. 10 are applicable.

Thus $\quad a^2+b^2+\ldots+k^2 = (a-l)^2+\ldots+(k-l)^2+nl^2$
$\qquad\qquad + 2l[(a-l)+\ldots+(k-l)]$

and $\quad nal+\frac{1}{3}n(a-l)^2 = nl(a-l)+nl^2+\frac{1}{3}n(a-l)^2$

But $\quad (a-l)^2+\ldots+(k-l)^2 > \frac{1}{3}n(a-l)^2$

and $\quad (a-l)+\ldots+(k-l) > \frac{1}{2}n(a-l)$

Hence $\quad a^2+b^2+\ldots+k^2 > nal+\frac{1}{3}n(a-l)^2$

and in the same way the other inequality is established.

The transformation here adopted brings the general A.P. within the range of his methods, and would have enabled him to sum the squares of any number of terms even when the least term is not equal to the common difference. It would, however, have been rather troublesome to work out the details and express the sum in a purely geometrical form, though Archimedes certainly shows remarkable skill in dealing with complicated cases like this.

7. Another extension of the theorem of §5 is needed for his cubatures in *Conoids and Spheroids*, and it is found in the 2nd proposition of that book.

Let $a, b, \ldots l$ be n lines in A.P. of which the common difference is l, and p any other line and let S denote the sum

$$(pa+a^2)+(pb+b^2+\ \ldots\ +(pl+l^2);$$

then the following inequalities hold, namely,

$$\frac{S}{n(pa+a^2)} > \frac{\frac{1}{2}p+\frac{1}{3}a}{p+a} > \frac{S-(pa+a^2)}{n(pa+a^2)} \qquad \text{(A)}$$

The proof is effected by considering separately the sums

$$p(a+b+\ldots+l) \quad \text{and} \quad a^2+b^2+\ldots+l^2$$

and applying to these the proper inequality theorems.

In order to make the obervations in the next section more easily understood, I will indicate the bearing of this theorem on the cubature of the hyperboloid of revolution. Suppose a segment cut off by a plane at right angles to the axis at distance a from the vertex; let the distance a be divided into n equal parts, the distances from the vertex of the points of section forming the A.P. $a, b, \ldots l$, and let the figures described in §4 be constructed. Then if p be the transverse axis of the hyperboloid, the bases of the cylinders forming the set C are proportional to

$$(pa+a^2), \quad (pb+b^2), \; . \; . \; . \; (pl+l^2)$$

and of those forming the set I, to

$$(pb+b^2), \; . \; . \; . \; (pl+l^2).$$

It is easy then to see that

$$\frac{C}{K} = \frac{S}{n(pa+a^2)} \quad \text{and} \quad \frac{I}{K} = \frac{S-(pa+a^2)}{n(pa+a^2)}$$

and the theorem of this article proves that

$$\frac{C}{K} > \frac{\frac{1}{2}p+\frac{1}{3}a}{p+a} > \frac{I}{K}$$

and the application of the method of exhaustion then shows that

$$\frac{\text{segment}}{K} = \frac{\frac{1}{2}p+\frac{1}{3}a}{p+a}$$

8. In the case of the ellipsoid of revolution the corresponding bases are proportional to

$$(pa-a^2), \quad (pb-b^2), \; . \; . \; . \; (pl-l^2)$$

and Zeuthen [*Kegelschnitte im Altertum*, p. 450] expresses surprise that Archimedes did not proceed in this case on the same lines as the above treatment of the hyperboloid. But it is not, I think,

hard to understand the difference of treatment; it is simply impossible by means of the inequalities alone to establish the proper relations for the ellipsoid. If σ represent the sum

$$(pa-a^2)+(pb-b^2)+\ \ldots\ +(pl-l^2)\,*$$

the relations required are

$$\frac{\sigma}{n(pa-a^2)}>\frac{\frac{1}{2}p-\frac{1}{3}a}{p-a}>\frac{\sigma-(pa-a^2)}{n(pa-a^2)}; \qquad \text{(B)}$$

but from the inequalities

$$p(a+b+\ldots+l)>\tfrac{1}{2}npa>p(b+\ldots+l)$$
$$a^2+b^2+\ldots+l^2)>\tfrac{1}{3}na^2>\ b^2+\ldots+l^2$$

it is only possible to conclude

$$pa+(pb-b^2)+\ldots+(pl-l^2)>na(\tfrac{1}{2}p-\tfrac{1}{3}a)$$

and

$$-a^2+(pb-b^2)+\ldots+(pl-l^2)<na(\tfrac{1}{2}p-\tfrac{1}{3}a)$$

or

$$\frac{\sigma+a^2}{n(pa-a^2)}>\frac{\frac{1}{2}p-\frac{1}{3}a}{p-a}>\frac{\sigma-ap}{n(pa-a^2)}$$

and this form is absolutely unsuitable. To get the proper form by this method, we have to take the exact value of σ, namely,

$$\sigma=na(\tfrac{1}{2}p-\tfrac{1}{3}a)+\tfrac{1}{2}a(p-a-\tfrac{1}{3}l)$$

and therefore

$$\sigma-(pa-a^2)=na(\tfrac{1}{2}p-\tfrac{1}{3}a)-\tfrac{1}{2}a(p-a+\tfrac{1}{3}l)$$

In order that (B) may be true, therefore, it is necessary to have $p>a+\frac{1}{3}l$, and though this condition is satisfied, it could not be established by means of the inequalities alone.

From formula (A) of §7 we get

$$lS>nla(\tfrac{1}{2}p+\tfrac{1}{3}a)>lS-l(pa+a^2)$$

Hence the limit of lS for $n=\infty$ (or $l=0$, since $nl=a$) is $\frac{1}{2}a^2p+\frac{1}{3}a$, so that the result is equivalent to the integration

$$\int_0^a (px+x^2)dx=\tfrac{1}{2}a^2p+\tfrac{1}{3}a^3$$

and in the same way the formula (B) is equivalent to

$$\int_0^a (px-x^2)dx=\tfrac{1}{2}a^2p-\tfrac{1}{3}a^3$$

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* In the diagram of Archimedes (*Opera* I., p. 462),
$p=\mathrm{BZ}$, $a=\mathrm{B}\Delta$, $b=\mathrm{BE}$ etc.

The fact, however, that Archimedes did not establish the theorem (B), which would have taken the place of the integral last written, seems to be due to his preference for inequalities, which in its turn was probably a consequence of his geometrical methods with their prolix enunciations rather than, as Zeuthen seems to think (p. 452), to the absence of a theorem corresponding to

$$\int[\phi(x)+\psi(x)]dx=\int\phi(x)dx+\int\psi(x)dx$$

for negative as well as positive values of $\psi\,(x)$. There is no doubt a great amount of truth in the general remarks of Zeuthen in the passage referred to, but the difficulty of establishing the inequality theorem (A) by a process equally applicable to theorem (B) or in general of establishing theorems that shall be equally applicable to positive and negative quantities is more than a difficulty of language. There is unquestionably a difficulty of language, but there is also a special difficulty arising from the use of inequalities, as in the case of theorem (B). The difference between ancient and modern methods introduced by the employment of negative quantities or negative operators seems to me to go deeper than is sometimes realised.

9. Had Archimedes first investigated the inequalities (B) he might have treated the cubature of the ellipsoid much more concisely. It may be noticed, however, that the transformation required in the case of a segment of the ellipsoid is at bottom identical with that of § 6. In dealing with the ellipsoid he requires to sum the series

$$u=(a^2-b^2)+(a^2-c^2)+\ .\ .\ .\ +(a^2-l^2)$$

where a, b, c,...l are $n+1$ lines in A.P. of which the common difference is not equal to the least line l. To effect the summation he puts

$$a^2-l^2=(a^2-x^2)+2l(x-l)+(x-l)^2$$

where x is any of the lines a, b, c, etc. Applying the theorem of § 7 to the series of n terms with

$$2l(x-l)+(x-l)^2$$

as general term, he gets

$$\frac{u}{n(a^2-l^2)}>\frac{2\ a+l}{3(a+l)}>\frac{u-(a^2-l^2)}{n(a^2-l^2)}$$

But since $$u = na^2 - (b^2 + \;.\;.\;.\; + l^2)$$
this pair of inequalities is equivalent to
$$a^2 + b^2 + \ldots + k^2 > \tfrac{1}{3}n(a^2 + al + l^2) > b^2 + \ldots + k^2 + l^2$$
and these are the inequalities established in § 6. *

10. Nearly all the theorems referred to in this paper seem to be due to Archimedes himself, and the whole treatment shows an originality of conception and execution that is somewhat difficult for us to recognise. The so-called geometrical algebra of the Greeks, valuable and important as it is for many purposes, is but a clumsy instrument compared with modern algebra in dealing with the summations discussed above, and in reading Archimedes one cannot fail to be struck with the prolixity of the enunciations and the length of the demonstrations caused in part by the absence of mere technical terms, but chiefly by the purely geometrical form in which his work is cast. It is, however, only an additional testimony to his genius that he triumphed over such difficulties and was able to carry the mensuration of the more common surfaces and solids to a stage which is even now the limit of instruction that does not involve the Integral Calculus.

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* As another illustration of the application of the inequality theorems, I had worked out the value of the area of a segment of a parabola from the figure used in the mechanical quadrature, but as the method I followed is identical with that given by Heath (p. cliv.), I omit my investigation.

[A a] [D 6 i] [H 12 b α] Le Quatrième Algorithme Naturel.

Par Monsieur E. M. LÉMERAY.

I. INTRODUCTION ET DÉFINITIONS.

On est amené à résoudre l'équation aux différences mêlées

$$y'(Cy-1) = \Delta y' - Cy'\Delta y$$

lorsqu'on cherche, en coordonnées rectangulaires, une courbe telle que : M et M′ étant deux de ses points dont les abscisses diffèrent d'une constante, la tangente de l'angle que fait avec OX la tangente en M soit en raison inverse de la sous-tangente en M′.

En posant $y + \Delta y = y_1$, l'équation peut s'écrire

$$(1)\quad y_1' = cy_1 y'$$

Considérons l'équation générale (2) $y_1'\ y_1^{n-1} = cy'$ qui se réduit à la première pour $n=0$; on peut en trouver une intégrale particulière. En intégrant sans constante arbitraire, on a :

$$y_1{}^n = ncy_0 \quad \text{puis} \quad y_2{}^n = ncy_1, \quad y_3{}^n = ncy_2 \ \ldots \ y_x{}^n = ncy_{x-1}$$

en éliminant $y_1\ y_2 \ldots y_{x-1}$ entre ces équations, on aura une fonction de la forme

$$y = AB^{\left(\frac{1}{n}\right)^x}$$

A et B étant des constantes arbitraires. Cette fonction définie pour des valeurs entières et positives de x, a cependant une signification quand x n'est pas un entier positif. Pour $n=1$ ou $n=0$ l'expression ci-dessus ne représente plus une fonction ; dans le premier cas l'équation (2) devient $y_1' = cy'$ et admet comme intégrale particulière la fonction exponentielle

$$y = AC^x$$

dans le second cas l'équation (2) se réduit à (1). Pour en trouver une intégrale particulière intégrons sans constante arbitraire.

On a $\qquad (3)\quad y_1 = a^{y_0} \quad y_2 = a^{y_1} \ \ldots \ y_x = a^{y_{x-1}}$

où l'on a posé $e^c = a$. Au lieu de la notation ordinaire p^q des puissances j'emploierai comme équivalente la notation plus commode $p\,.\,q$. Cela n'a pas d'inconvénient car je ne ferai pas usage du point

entre deux lettres pour représenter une multiplication ; il faudra seulement se rappeler que l'on n'a pas $p.q = q.p$. Eliminons $y_1, \; y_2, \ldots y_{x-1}$ entre les équations (3) ; on aura

$$y = a.a.a. \quad . \quad . \quad . \quad a.y_0$$

où a est écrit x fois et que je noterai $\begin{array}{r|l} x & y_0 \\ a & \end{array}$

Cette fonction est intéressante en ce que l'algorithme qui sert à l'exprimer peut être considéré comme un algorithme naturel. Je veux dire par là qu'on peut trouver une loi faisant dériver l'un de l'autre les trois algorithmes naturels, addition, multiplication, élévation aux puissances, et faisant dériver de ce dernier, l'algorithme dont il est question. En effet de même que la substitution $y_0, \; a+y_0$ effectuée x fois fournit la fonction $y = ax + y_0$, et que la substitution $y_0, \; ay_0$ effectuée x fois fournit la fonction $y = a^x \times y_0$; de même la substitution $y_0, \; a^{y_0}$ effectuée x fois fournit la fonction $y = \begin{array}{r|l} x & y_0 \\ \dot{a} & \end{array}$

On pourrait évidemment continuer ainsi, et engendrer une suite illimitée d'algorithmes naturels directs.

L'idée d'exponentielles superposées n'est pas nouvelle ; ainsi l'on a énoncé cette proposition que les nombres de la suite

$$2+1 \qquad 2^2+1 \qquad 2^{2^2}+1 \; . \; . \; .$$

sont premiers. Dans la " Théorie des nombres et Algèbre supérieure " de MM. Borel et Drach, d'après les leçons de M. Tannery, M. Drach dit : " Remarquons . . . . qu'il serait possible de continuer dans la voie où nous nous sommes engagés, et qu'il serait, par exemple, utile d'introduire une nouvelle notation pour représenter les nombres $a^b \; a^{b^b} \; . \; . \; .$ en mettant en évidence les rôles que jouent a et b, et le nombre des exposants superposés " (Mes premieres études à ce sujet ont été publiées avant l'ouvrage de MM. Borel et Drach) Mais on n'a pas, du moins à ma connaissance, considéré ce nouvel algorithme dans lequel la lettre qui diffère des autres est la lettre supérieure au lieu d'être la base. La distinction est loin d'être sans importance, notre point de vue est au contraire nécessaire pour établir un théorème d'addition et les théorèmes subséquents. Pour éviter de longues périphrases j'emploierai trois néologismes, et aussi des noms de fonctions usuelles

en leur adjoignant une lettre spéciale pour désigner les nouvelles fonctions. C'est ainsi que je dénommerai le nouvel algorithme sous le nom de surpuissance, et dans l'expression :

$$p = \overset{b}{\dot{a}}\,\Big|^{q}$$

j'appellerai a la base de la surpuissance, b son exposant, c l'exposant initial ou plus simplement l'initial.

Dans cette expression nons pouvons considérer deux lettres comme constantes, une comme variable, et la quatrième comme fonction :

p considéré comme fonction de a, b et q étant constants, est la surpuissance,
p considéré comme fonction de b, a et q étant constants, sera la fonction surexponentielle :

Il n'y a pas lieu de considérer p comme fonction de q ; cet algorithme donne naissance à deux fonctions inverses ; nous dirons que a considéré comme fonction de p est la surracine $b^{\text{ième}}$ de p, et que b considéré comme fonction de p est son hyperlogarithme dans le système de base a ; nous adopterons pour ces deux fonctions les symboles

$$a = \sqrt[b]{p} \qquad b = \mathrm{HL}_a p$$

Je considérerai encore d'autres fonctions. Dans l'expression $p = \overset{b}{\dot{a}}\,\Big|^{q}$ posons $\overset{b}{\dot{a}} = u$, nous regarderons comme équivalente la notation

$$p = u\,\Big|^{q}$$

en sous-entendant que la base est a. Autrement l'expression n'aurait pas de sens et pour expliciter q en fonction de p et de u, nons écrirons

$$q = {}^{p}\Big|_{u}$$

L'expression

$$p = u\,|^{u}\,|^{u}$$

où le nombre des u est m sera la puissance-p $m^{\text{ième}}$ de u, nous noterons cette fonction par le symbole

$$p = u^{\widehat{m}}$$

Inversement u sera la racine-r $m^{\text{ième}}$ de p et l'on écrira

$$u = \overset{m}{\mathrm{W}}\overline{p}$$

Enfin m considéré comme fonction de p sera son logarithme $-\ l$, dans le système de base u; la base de la surpuissance est toujours sous-entendue être a. On voit de plus que si $u = a$, la puissance $-\ p$ peut s'écrire $a\,.\,a\,.\,a$ et se réduit ä une surpuissance. Je représenterai le logarithme $-\ l$ par le symbole $m = \mathrm{AL}_u p$.

J'ai d'abord étudié les propriétés de ces nouveaux algorithmes sans tenir compte du mode de génération par substitutions uniformes; puis j' ai cherché les propriétés analogues en tenant compte du mode de génération et sans faire d'hypothèses sur le premier algorithme dont on part; enfin en supposant que ce premier algorithme est l'addition, j'ai appliqué les lois générales précédemment trouvées, et j'ai retrouvé comme cas particuliers des théorèmes comme sur les trois premiers algorithmes naturels et ceux que j'avais démontrés isolément sur la surpuissance. La place dont je dispose étant mesurée, je ne pourrai, à mon regret, donner l'étude générale dont je viens de parler; je donnerai seulement les théorèmes particuliers au nouvel algorithme; mais à côté et entre parenthèses je rappellerai les théorèmes analogues relatifs au troisième algorithme.

Théorèmes d'addition et thèorèmes subséquents.

De la définition même il résulte que l'on a

$$\left.\begin{matrix} b+b_1 \\ \dot{a} \end{matrix}\right|^{c} = \left.\begin{matrix} b \\ \dot{a} \end{matrix}\right|^{\left.\begin{matrix} b \\ \dot{a} \end{matrix}\right|^{c}} \qquad \left[a^{b+b_1} \times c = a^{b} \times a^{b_1} \times c\right]$$

c'est l'expression $\left.\begin{matrix} b \\ \dot{a} \end{matrix}\right|^{c}$ dans laquelle l'initial a été remplacé par $\left.\begin{matrix} b_1 \\ \dot{a} \end{matrix}\right|^{c}$.

Exposant différence. On a aussi

$$\left.\begin{matrix} b-1 \\ \dot{a} \end{matrix}\right|^{c} = \log_a \left.\begin{matrix} b \\ \dot{a} \end{matrix}\right|^{c}, \qquad \left.\begin{matrix} b-2 \\ \dot{a} \end{matrix}\right|^{c} = \log_a \left.\begin{matrix} b-1 \\ \dot{a} \end{matrix}\right|^{c} \ \ldots$$

et en général

$$\left.\begin{matrix} b-b_1 \\ \dot{a} \end{matrix}\right|^{c} = \log_a^{(b_1)} \left.\begin{matrix} b \\ \dot{a} \end{matrix}\right|^{c} \qquad \left[a^{b-b_1} \times c = \left[\left[a^{b} \times c\right] : a\right] : a \ \ldots\right]$$

le symbole $\log_a^{(b_1)}$ représentant l'opération qui consiste à prendre b_1 fois de suite dans le système de base a, le logarithme de $\left.\begin{matrix} b \\ \dot{a} \end{matrix}\right|^{c}$

Exposant zéro. En faisant $b_1 = b$ on a

$$\overset{0}{\dot{a}}\Big|^{c} = \overset{b-b_1}{\dot{a}}\Big|^{c} \quad \log_a^{(b)} \overset{b}{\dot{a}}\Big|^{c} = c \qquad \left[a^{0} \times c = c\right]$$

la surpuissance d'exposant nul est donc égale à son initial.

Exposant négatif. Si b_1 est plus grand que b et égal à $b + b_2$ on a :

$$\overset{-b_2}{\dot{a}}\Big|^{c} = \log_a^{(b_1)} \overset{b}{\dot{a}}\Big|^{c} = \log_a^{(b_1-b)} \overset{0}{\dot{a}}\Big|^{c} = \log_a^{b_2} c \qquad \left[a^{-b_2} \times c = \frac{c}{a^{b}}\right]$$

Si l'on fait $b_2 = 1$ on a :

$$\overset{-1}{\dot{a}}\Big|^{c} = \log_a c \qquad \left[a^{-1} \times c = \frac{c}{a}\right]$$

La comparaison de ces deux relations nous montre que dans l'algorithme de l'élévation aux puissances, l'opération analogue à la division de c par a n'est pas l'extraction d'une racine, mais bien l'opération qui consiste à prendre, dans le système de base a, le logarithme de c.

Permutation de la fonction et de l'initial. Soit la relation

$$p = \overset{-b}{\dot{a}}\Big|^{q}$$

prenons les deux membres de cette égalité pour en faire les initiaux d'une surpuissance de base a et d'exposant b, nous aurons encore une égalité

$$\overset{b}{\dot{a}}\Big|^{p} = \overset{b}{\dot{a}}\Big|^{\overset{-b}{\dot{a}}\big|^{q}} = \overset{b-b}{a}\Big|^{q} = \overset{0}{\dot{a}}\Big|^{q} = q$$

il y a donc équivalence entre les égalités

$$p = \overset{-b}{\dot{a}}\Big|^{q}, \quad q = \overset{b}{\dot{a}}\Big|^{p} \qquad \left[p = a^{b} \times q, \quad q = a^{-b} \times p\right]$$

on peut donc permuter le prémier membre et l'initial du second pourvu qu'on change le signe de l'exposant.

Exposant produit. On a évidemment

$$\overset{mb}{\dot{a}}\Big|^{c} = \overset{b}{\dot{a}}\Big|^{\overset{b}{\dot{a}}\big|^{\cdot^{\cdot^{\cdot}}\big|^{c}}} = \left(\overset{b}{\dot{a}}\right)^{\widehat{m},\ c} \qquad \left[a^{mb} \times c = \left(a^{b}\right)^{m} \times c\right]$$

c'est donc une puissance $-p$; on a évidemment

$$\left(\begin{matrix} b \\ \dot{a} \end{matrix}\right)^{\widehat{m},\, c} = \begin{matrix} mb \\ \dot{a} \end{matrix}\Big|^{c} = \left(\begin{matrix} m \\ \dot{a} \end{matrix}\right)^{\widehat{b},\, c} \qquad \left[\left(\frac{b}{a}\right)^{m} \times c = \left(\frac{m}{a}\right)^{b} \times c\right]$$

On peut donc permuter l'exposant de la puissance $-p$ et celui de la surpuissance. On voit aussi que dans le troisième algorithme la puissance est à la fois l'analogue de la surpuissance et l'analogue de la puissance $-p$.

Exposant quotient. Dans la relation précédente posons $mb = p$ d'où $b = \dfrac{p}{m}$ on aura

$$\begin{matrix} p \\ \dot{a} \end{matrix}\Big|^{c} = \left(\begin{matrix} \frac{p}{m} \\ \dot{a} \end{matrix}\right)^{\widehat{m},\, c}$$

$\begin{matrix} \frac{p}{m} \\ \dot{a} \end{matrix}$ est donc a racine $-r$ $m^{\text{ième}}$ de $\begin{matrix} p \\ \dot{a} \end{matrix}\Big|^{c}$ et l'on a

$$\begin{matrix} \frac{p}{m} \\ \dot{a} \end{matrix}\Big|^{c} = \overset{m}{\mathrm{W}}\overline{\begin{matrix} b \\ \dot{a} \end{matrix}}\Big|^{c} \qquad \left[a^{\frac{p}{m}} \times c = \sqrt[m]{a^{p}} \times c\right]$$

Comme dans tout ce qui précède l'exposant de la surpuissance est supposé entier, et alors toutes ces expressions peuvent être calculées ; dans le cas contraire, on éprouve les mêmes difficultés pour vérifier que la valeur trouvée pour lá racine $-r$ répond à la question, que pour vérifier que la racine $n^{\text{ième}}$ d'un nombre qui n'est pas une puissance $n^{\text{ième}}$ exacte élevée à la $n^{\text{ième}}$ puissance reproduit le nombre donné ; c'est par une extension aux nombres quelconques de lois établies pour des nombres entiers que la vérification peut se faire. Par la comparaison des deux dernières relations on voit que dans le troisième algorithme l'analogue de lá racine $-r$ est la racine ; comme celleci elle s'exprime par l'exposant fractionnaire.

L'algorithme $u\Big|^{v}$.—La base du système de surpuissances étant choisie une fois pour toutes et sous-entendue ; on a

$$u\Big|^{v} = v\Big|^{u}$$

en supposant $u = \begin{matrix} b \\ \dot{a} \end{matrix}$ $v = \begin{matrix} b_1 \\ \dot{a} \end{matrix}$; comme la multiplication cette opération est commutative ; mais la différence consiste en ce que le produit

$u \times v = v \times u$ conserve la même valeur quelle que soit la base des puissances ; c'est à dire que u et v soient respectivement égaux à $\overset{b}{a\times}$ et à $\overset{b_1}{a}$ ou bien à $\overset{d}{c}$ et $\overset{d_1}{c}$, tandis que $u\big|^{v}$ change de valeur quand on change la base des surpuissances. La même restriction est applicable aux théoremes suivants.

Hyperlogarithmes. D'après les definitions précédentes on a d'une manière presque evidente

$$\mathrm{HL}u\,\big|^{v} = \mathrm{HL}v\,\big|^{u} = \mathrm{HL}u + \mathrm{HL}v \qquad \Big[\mathrm{Log}(u \times v) = \mathrm{Log}u + \mathrm{Log}v\Big]$$

on a aussi

$$\mathrm{HL}^{u}\big|v = \mathrm{HL}u - \mathrm{HL}v \qquad \Big[\mathrm{Log}\ \frac{u}{v} = \mathrm{Log}u - \mathrm{Log}v\Big]$$

On verrait facilement que si l'on a

$$p = u\,\big|^{v} \qquad [p = u \times v]$$

on en tire

$$v = {}^{p}\big|u \quad u = {}^{p}\big|v \qquad \Big[v = \frac{p}{u} \quad u = \frac{p}{v}\Big]$$

Hyperlogarithme d'une puissance $-p$ et d'une racine $-r$

On a presque évidemment

$$\mathrm{HL}u^{\widehat{m}} = m\ \mathrm{HL}u \qquad \Big[\ \mathrm{Log}\,u^{m} = m\mathrm{Log}u\ \Big]$$

$$\mathrm{HL}u^{\widehat{\frac{1}{m}}} = \frac{1}{m}\ \mathrm{HL}u \qquad \Big[\ \mathrm{Log}\sqrt[m]{u} = \frac{1}{m}\mathrm{Log}u\ \Big]$$

Logarithmes $-\,l$. Soit :

$$p = u^{\widehat{m}} = u_1^{\widehat{m_1}}$$

Si l'on suppose $u = \overset{b}{\dot{a}}$ $u_1 = \overset{b_1}{\dot{a}}$ on a :

$$p = \overset{bm}{\dot{a}} = \overset{b_1m_1}{\dot{a}} \quad \text{d'où} \quad bm = b_1m_1 \quad \text{et} \quad \frac{m}{m_1} = \frac{b_1}{b}$$

comme d'autre part, on a :

$$b = \mathrm{HL}_a u \qquad b_1 = \mathrm{HL}_a u_1 \qquad m = \mathrm{AL}_u p \qquad m_1 = \mathrm{AD}_{u_1} p$$

on a

$$\mathrm{AL}_{u_1} p = \mathrm{AL}_u p \frac{\mathrm{HL}_a u}{\mathrm{HL}_a u_1} \qquad \Big[\log_{u_1} p = \log_u p \frac{\log_a u}{\log_a u_1}\Big]$$

Donc pour passer d'un système de logarithmes $-l$ dans un autre, il faut multiplier les logarithmes $-l$ du premier système, par le rapport des hyperlogarithmes de la base du premier et de la base du second. Dans le troisième algorithme naturel, le logarithme est donc l'analogue à la fois de l'hyperlogarithmes et du logarithme $-l$. Mais $\dfrac{\log_a u}{\log_a u_1}$ est indépendant de a, ce qui n'est pas pour les hyperlogarithmes.

II. Expression de quelques fonctions trancendantes.

En analyse, on définit un certain nombre d'algorithmes directs, l'addition la multiplication, l'élévation aux puissances, la dérivation, les puissances d'une substitution, etc. . . . ; et quand on a pu mettre un problème quelconque en équations, on se trouve en présence d'un système de m équations entre $m+n$ variables. Ces équations peuvent contenir ou non des dérivées, des différences finies des puissances de fonctions, etc. . . . Mais on peut arriver à ce qu'il n'y entre que les symboles des algorithmes directs définis précédemment. Parmi les $m+n$ variables désignons n d'entre elles par $x_1\ x_2 \ldots x_n$ et les m autres par $y_1\ y_2 \ldots y_m$. Résoudre le système par rapport aux y, c'est arriver antant que possible à remplacer le premier système par un système équivalent composé de m équations telles que le premier membre de la $i^{\text{ème}}$ équation se réduise à y_i et le deuxième membre à $F_i\,(x_1\ x_2 \ldots x_n)$; la fonction F_i ne contenant que les x et des constantes finies et n'étant construite qu'avec les symboles directs des algorithmes naturels. On dira qu'on a exprimé explicitement les y au moyen des algorithmes naturels directs. Si l'on ne peut obtenir une résolution de cette nature et s'il est nécessaire d'employer certains symboles P, Q, R . . de fonctions inverses, on pourra dire, qu'on a explicité les y au moyen des algorithmes directs naturels avec adjonction des symboles P, Q, R . . . Les expressions obtennes n'auront de valeur que si l'on connait bien les fonctions inverses représentées par les symboles adjoints. Par exemple l'équation $x=a^y$ ne peut être résolue par rapport à y qu'avec adjonction du symbole et la soustraction et du symbole du logarithme

$$y=\log x \times (\log a)^{-1}.$$

Il peut aussi se présenter le cas où l'expression explicite des y_i peut s'obtenir à condition que certaines constantes soient infinies ;

ainsi la racine réelle de l'équation que nous avons prise pour exemple est la limite de

$$\frac{x^{\frac{1}{m}} - 1}{a^{\frac{1}{m}} - 1} \quad \text{(pour } m \text{ infini)}$$

et celle-ci, après qu'on aura fait disparaitre le symbole inverse division en l'exprimant par la puissance − 1, ne contiendra plus que les algorithmes naturels, le symbole inverse—mais elle contiendra l'infini. Dans d'autres cas on ne pourra mettre les y_i sous de telles formes, et on sera obligé de se contenter d'une résolution moins complète ; s'il s'agit par exemple d'équations différentielles et qu'on ait ramené, le problème à des quadratures, on aura explicité non plus la fonction mais bien une de ses dérivées ; enfin l'on pourra faute de mieux représenter la fonction inconnue par un développement, série, produit, etc. . . . et l'on arrivera à une notation qui pourra être finie mais au moyen du calcul symbolique. Si l'on a par exemple un développement en série de Maclaurin on pourra écrire

$$y_i = e^{\phi x}$$

en remplaçant après développement ϕ^k par $\left(\frac{d^k y_1'}{dx^k}\right)_0$; la formule n'aura de valeur que si l'on peut ensuite exprimer $\left(\frac{d^k y_1'}{dx^k}\right)_0$ en fonction de K soit en termes finis soit par une formule symbolique et ainsi de suite. Je n'ai pas d'ailleurs à insister sur ce sujet, les remarques que je vieus de rappeler n'ont d'autre but que de montrer l'intérêt qu'il y a à employer l'algorithme naturel de la surpuissance car par son moyen l'on peut expliciter certaines fonctions nouvelles, soit en termes finis, soit comme limites d'expressions directes quand certaines constantes dont elles dépendent tendent vers l'infini.

Expression finie du logarithme—Nous avons vu plus haut que l'on a

$$\log_a c = \dot{a}^{-1} \Big| c$$

Cette équivalence est tout à fait comparable à la suivante

$$\frac{c}{a} = a^{-1} \times c$$

tant qu'on ne possédait pas l'algorithme de l'élévation aux puissances, on ne pouvait expliciter au moyen des algorithmes naturels directs la racine de l'équation $\qquad ax = c$

De même ne possédant pas l'algorithme de la surpuissance, on ne peut actuellement expliciter la racine (ou plutôt les racines) de l'équation $a^x = c$. L'algorithme de la surpuissance atteint ce but.

La loi est d'ailleurs générale ; si l'on a

$$\overset{\overset{x}{\cdot}}{a} = c$$

on aura $\qquad HL_a c = \left.\overset{\overset{-1}{\cdot\cdot}}{a}\,\right\|^{c}$

ce symbole représentant le cinquième algorithme naturel. Bien entendu, pas plus que $a^{-1} \times c$, l'expression $\left.\overset{\overset{-1}{\cdot}}{a}\,\right|^{c}$ ne représente les *calculs* qu'il faut faire sur a et c pour obtenir la valeur *numérique* de x.

Il résulte de là que les fonctions circulaires ou hyperboliques inverses pourront s'expliciter au moyen des algorithmes naturels avec adjonction du seul symbole inverse.—On sait en effet qu'elles se ramènent à des logarithmes népériens de quantités imaginaires ou réelles. Ainsi l'on a :

$$\text{arc tang}\, x = \frac{L(1 + x\sqrt{-1}) - L(1 - x\sqrt{-1})}{2\sqrt{-1}}$$

on pourra par suite écrire

$$\text{arc tang}\, x = \frac{\left.\overset{\overset{-1}{\cdot}}{e}\,\right|^{1+x\sqrt{-1}} - \left.\overset{\overset{-1}{\cdot}}{e}\,\right|^{1-x\sqrt{-1}}}{2\sqrt{-1}}$$

formule qui présente la même symétrie que celle qu'Euler a donnée pour $\sin x$. L'algorithme de la surpuissance permet de faire certains rapprochements curieux. Les expressions générales

$$\frac{\left.\overset{\overset{m}{\cdot}}{e}\,\right|^{1+x\sqrt{-1}} + \left.\overset{\overset{m}{\cdot}}{e}\,\right|^{1-x\sqrt{-1}}}{2} \qquad\qquad \frac{\left.\overset{\overset{m}{\cdot}}{e}\,\right|^{1+x\sqrt{-1}} - \left.\overset{\overset{m}{\cdot}}{e}\,\right|^{1-x\sqrt{-1}}}{2\sqrt{-1}}$$

donnent respectivement, quand on y fait $m = -1$, $m = 0$, $m = 1$, les fonctions

$$L\sqrt{1 + x^2}, \quad 1, \quad e\cos x \qquad\qquad \text{arc tang}\, x, \quad x, \quad e\sin x$$

qui sont ainsi des cas particuliers d'une fonction plus générale.

Les identités

$$a^x \times a^y = a^{x+y} \qquad a^x : a^y = a^{x-y}$$

$$\log_a x + \log_a y = \log_a (x \times y) \qquad \log_a x - \log_a y = \log_a \frac{x}{y}$$

pourront s'écrire

$$\begin{array}{c}1\\ \dot{a}\end{array}\Bigg|^{x}\begin{array}{c}1\\ \times\,\dot{a}\end{array}\Bigg|^{y}\begin{array}{c}1\\ =\dot{a}\end{array}\Bigg|^{x+y} \qquad \begin{array}{c}1\\ \dot{a}\end{array}\Bigg|^{x}\begin{array}{c}1\\ :\,\dot{a}\end{array}\Bigg|^{y}\begin{array}{c}1\\ =\dot{a}\end{array}\Bigg|^{x-y}$$

$$\begin{array}{c}-1\\ \dot{a}\end{array}\Bigg|^{x}\begin{array}{c}-1\\ +\ \dot{a}\end{array}\Bigg|^{y}\begin{array}{c}-1\\ =\ \dot{a}\end{array}\Bigg|^{x\times y} \qquad \begin{array}{c}-1\\ \dot{a}\end{array}\Bigg|^{x}\begin{array}{c}-1\\ -\ \dot{a}\end{array}\Bigg|^{y}\begin{array}{c}-1\\ =\ \dot{a}\end{array}\Bigg|^{x:y}$$

et dans les quatre cas l'exposant de la surpuissance est égal en grandeur et en signe à l'excès de l'ordre de l'algorithme qui relie l'une à l'autre les deux fonctions dans le premier membre, sur l'ordre de l'algorithme qui relie x à y dans l'initial du second, que ces algorithmes soient directs ou inverses.

III. Expression directe des racines des équations $x = a^{x}.\ \overset{2}{\dot{x}} = a.$

Je me propose de montrer que les racines de ces équations peuvent s'exprimer explicitement comme limites des valeurs que prennent certaines expressions construites seulement au moyen des algorithmes naturels avec adjonction du seul symbole inverse—. Occupons nous d'abord des racines réelles. Construisons en coordonnées rectangulaires les courbes figuratives des fonctions $y = x$, $y = a^x$ leurs intersections auront pour abscisses les racines réelles de l'équation $x = a^x$. Une discussion facile montrerait que l'on obtient :

pour	$a > e^{\frac{1}{e}}$	aucune racine réelle
	$a = e^{\frac{1}{e}}$	une racine double égale à e
	$1 < a < e^{\frac{1}{e}}$	deux racines comprises l'une entre 1 et e, l'autre entre e et ∞
	$1 > a > \left(\frac{1}{e}\right)^e$	une racine réelle comprise entre $\frac{1}{e}$ et 1
	$a = \left(\frac{1}{e}\right)^e$	une racine simple égale à $\frac{1}{e}$
	$\left(\frac{1}{e}\right)^e > a > 0$	une racine réelle comprise entre 0 et $\frac{1}{e}$

J'ai déja démontré * les résultats que je vais rappeler ici. Pour ne

* *Nouvelles Annales* (*Laisant et Antomari*), Décembre 1896, Février 1897.

pas allonger cette note je ne donnerai pas ces démonstrations ; mais je les exposerai au moyen d'une représentation géométrique qui aura l'avantage de parler aux yeux.

FIGURE 1.

Considérons d'abord le cas $1 < a < e^{\frac{1}{e}}$ auquel se rapporte la Fig. 1 ; les deux racines sont u et U. Soit E le point de la bissectrice des axes ayant une ordonnée égale à e. Suivons le chemin brisé rectangulaire E A B C . . . dont les sommets sont alternativement situés sur la droit $y = x$ et sur la courbe $y = a^x$; calculons les ordonnées des points E B D . . . Pour B l'on a

$$\text{ordonnée B} = \text{ordonnée A} = a^e = a\,.\,e = \left.\overset{1}{\dot{a}}\right|^{e}$$

$$\text{ordonnée D} = \text{ordonnée C} = a\,.\,a\,.\,e = \left.\overset{2}{\dot{a}}\right|^{e} \quad \ldots$$

ces différents points représentant par leurs ordonnées les valeurs successives de $\left.\overset{m}{\dot{a}}\right|^{e}$ quand m prend les valeurs entières 1, 2, 3 . . . Or l'on *voit* que ces points convergent vers l'intersection dont l'abscisse est u ; on a ainsi

$$u = \lim \left.\overset{m}{\dot{a}}\right|^{e} \quad \text{pour } m \text{ infini.}$$

Si maintenant l'on suit le chemin E $A_1B_1C_1$ . . . on a :

$$\text{ordonnée } B_1 = \log_a \text{ ordonnée } A_1 = \left.\overset{-1}{\dot{a}}\right|^{e}$$

$$\text{ordonnée } D_1 = \log_a \text{ ordonnée } C_1 = \log_a \text{ ordonnée de } B_1 = \left.\overset{-2}{\dot{a}}\right|^{e} \quad \ldots$$

les points B_1 D_1 . . . convergent vers l'intersection d'abscisse U, et l'on a

$$U = \lim \left.\overset{-m}{\dot{a}}\right|^{e}$$

FIGURE 2.

Considérons maintenant le cas $1 > a > \left(\frac{1}{e}\right)^{e}$. Un calcul simple montrerait que au point d'intersection a de la bissectrice avec $y = a^x$ la dérivèe a une valeur comprise entre o et -1. Soit E le point de $y = x$ ayant pour abscisse $\frac{1}{e}$. Suivons le chemin

E A B C . . . défini comme précédemment, on a,

$$\text{ordonnée B} = \text{ordonnée A} = a^{\frac{1}{e}} = a \,.\, \frac{1}{e}$$

$$\text{ordonnée D} = \text{ordonnée C} = a \,.\, a \,.\, \frac{1}{e}, \quad \text{etc.} \ldots$$

On voit que les point B, D, . . . situés de par et d'autre de l'intersection s'en approchent de plus en plus ; on a ainsi :

$$a = \lim \overset{m}{\dot{a}} \Big|^{e^{-1}} \quad \text{(pour } m \text{ infini)}$$

Considérons enfin le cas où l'on a $\left(\frac{1}{e}\right)^e > a > 0$. On verrait facilement qu'alors la dérivée $\frac{da^x}{dx}$ a au point d'intersection une valeur comprise entre -1 et $-\infty$

FIGURE 3.

E ayant encore pour abscisse $\frac{1}{e}$, on a sur le chemin EABC . . .

$$\text{ordonnée B} = \log_a \text{ ordonnée A} = \overset{-1}{\dot{a}} \Big|^{\frac{1}{e}}$$

$$\text{ordonnée D} = \log_a \text{ ordonnée C} = \overset{-2}{\dot{a}} \Big|^{e^{-1}} \quad \text{etc.} \ldots$$

les points B D se rapprochent du point d'intersection et l'on a

$$a = \lim \overset{-m}{\dot{a}} \Big|^{e^{-1}}$$

Il est clair qu'au lieu de partir des points E on aurait pu partir d'autres points de la bissectrice des axes ; ainsi dans le premier cas on pouvait partir d'un point quelconque compris entre les deux intersections, mais celles-ci sont supposées inconnues et les valeurs e et $\frac{1}{e}$ attribuées à l'initial sont les seules qui conviennent *toujours*. En résumé l'on a l'expression générale des racines réelles

$$\lim \overset{\pm m}{\dot{a}} \Big|^{e^{\pm 1}}$$

avec les quatre combinaisons de signes suivantes :

Pour $e^{\frac{1}{e}} > a > 1$ on a deux racines répondant aux combinaisons $\begin{cases} + + \\ - + \end{cases}$

$1 > a > \left(\frac{1}{e}\right)^e$ une racine répondant à la combinaison $+ -$

$\left(\frac{1}{e}\right)^e > a > 0$ une racine répondant à la combinaison $- -$

Passons maintenant aux racines imaginaires. Notre équation revient à

$$a^{u+v\sqrt{-1}} = u + v\sqrt{-1}$$

où $u + v\sqrt{-1}$ est une racine ; elle équivaut aux deux suivantes

$$(1) \qquad u = a^u \cos(vLa) \qquad v = a^u \sin(vLa)$$

dans l'une on peut expliciter v, dans l'autre u

$$v = \frac{1}{La} \text{arc cos} \frac{u}{a^u} \qquad u = \frac{1}{La} L \frac{v}{\sin(vLa)}$$

on peut donc construire ces courbes dont chaque intersection correspondra à une racine de la proposée. Cela n'offre aucune difficulté et l'on verrait que si a est plus grand que 1, on trouve une intersection et une seule dans l'intervalle compris entre les droites $v = \frac{2K\pi}{La}$ et $v = \frac{(2K+1)\pi}{La}$. Rappelons un théorème connu : Si $f(x)$ désigne une fonction holomorphe dans le voisinage d'un point a supposé être un point racine de l'équation $f(x) - x = 0$ et si le module de la dérivée $\frac{dfx}{dx}$ prend, pour $x = a$, une valeur inférieure à 1 ; la substitution i, $f(i)$ tendra vers une limite égale à a ; pourvu que i soit pris dans une région convenable. Si le module de la dérivée est plus grand que 1, et si l'on sait inverser la fonction $f(x)$; la substitution i, $f^{-1}i$ convergera vers a à condition qu'on ne conserve que les déterminations convenables. (Ce théorème aurait pu nous servir dans le cas des racines réelles.)

Dans notre cas il est facile le voir qu'en un point racine quelconque le module de la dérivée $\frac{da^x}{dx}$ est toujours plus grand que 1. Posons

$$(2) \qquad u = \rho\cos\theta \qquad v = \rho\sin\theta$$

le module cherehé est $\quad a^u La = \rho La = M$

Comme en un point racine quelconque on doit vérifier les équations

$$\rho = a^u \qquad \theta = v\mathrm{L}a$$

qu'on déduit en combinant les systèmes (1) et (2), on en tire $\mathrm{M} = \frac{\theta}{\sin\theta}$ rapport toujours supérieur à 1, sauf pour $\theta = 0$ (racines réelles).
La substitution $x,\ a^x$ ne convergera donc jamais vers a quel que soit l'initial ; mais sous la restriction mentionnée la substitution inverse $x,\ \left.\overset{-1}{\dot{a}}\right|^{x}$ $\left(\text{c'est à dire dans les notations usuelles } x,\ \frac{\mathrm{Log}x}{\mathrm{Log}a}\right)$ convergera vers a.

Soit $\rho_0 e^{\theta_0\sqrt{-1}}$ une valeur quelconque fournie par la substitution ; cherchons à figurer celle qui est fournie par la substitution répétée une fois de plus.

Si $u_1 + v_1\sqrt{-1}$ est cette valeur on devra avoir

$$\text{(A)} \qquad \mathrm{L}\rho_0 = u_1\mathrm{L}a \qquad \theta_0 = v_1\mathrm{L}a$$

on peut donc calculer u_1 et v_1 puis ρ_1 et θ; ce calcul correspond à la construction suivante. Dans le système de coordonnées rectangulaires $u,\ v$ traçons les courbes $v = a^u$, $u = \cos(v\mathrm{L}a)$ et le cercle $\rho = 1$.

Figure 4.

Soit p l'affixe de $u_0 + v_0\sqrt{-1}$; suivons les chemins $p\mathrm{LM}z$ et $p\mathrm{L}_1\mathrm{M}_1z_1$ formés : le premier d'un arc de cercle de centre O, d'une paralèlle à ou, et d'une parallèle à ov; le second d'une droite passant à l'origine, d'une parallèle à ov, et d'une parallèle à ou; les derniers éléments de ces chemins se coupent en q, ce point est l'affixe de $u_1 + v_1\sqrt{-1}$. En effet l'on a

$$\text{absc.}q = \text{absc.M} = \log_a(\text{ord.M}) = \log_a \overline{\mathrm{OL}} = \log_a op = \log_a\rho_0$$

$$\text{ord. } q = \text{ord. } \mathrm{M}_1 = \frac{1}{\mathrm{L}a}\,\text{arc}(\cos = \text{absc.M}_1) = \frac{1}{\mathrm{L}a}\text{arc}(\cos = \text{absc.L}_1 = \frac{1}{\mathrm{L}a}\theta_0$$

Pour abréger je dirai que q est le logarithme le p. Cela posé, prenons pour initial la quantité $\sqrt{-1}$. En calculant $u_1 + v_1\sqrt{-1}$ ou en le construisant géométriquement on obtiendra tous les points situés sur l'axe ov et ayant pour ordonnées $(4j \pm 1)\frac{\pi}{2\mathrm{L}a}$, j étant

entier. Si l'on veut obtenir la racine a_k comprise entre les droites $\frac{2k\pi}{La}$ et $2(k+1)\frac{\pi}{La}$ il faut avoir

$$4k < 4j \pm 1 < 4(k+1)$$

il faudra donc faire $j=k$ et ne pas tenir compte du signe − qui fournirait des racines étrangères. Soit R_1 le point obtenu ; en continuant ainsi, on obtiendra un point R_2 comme logarithme de R_1, et qui sera situé à l'intersection de la droite $v=(4k+1)\frac{\pi}{2La}$ et de la courbe $v=a^u$, et ainsi de suite. En vertu du théorème rappelé plus haut on convergera vers a_k ; j'ai cru bon d'indiquer cette représentation géométrique qui, si on la dessine complètement fera voir que les points R_1, R_2, . . . ainsi obtenus s'approchent de plus en plus du point R intersection des courbes definies par les équations (A), et supposées tracées. En résumé, sous la restriction mentionnée plus haut, les racines imaginaires seront les limites de l'expression

$$\overset{-m}{\dot{a}}\Big|^{\pm\sqrt{-1}} \qquad \text{(pour } m \text{ infini)}$$

Quand a est plus petit que 1 et égal à $\frac{1}{b}$; les mêmes considérations sont applicables à part cette différence que les racines de la proposée sont comprises entre les droites

$$v=(2k-1)\frac{\pi}{La} \quad \text{et} \quad v=2\frac{k\pi}{La} ;$$

il faudra alors dans le facteur $4j \pm 1$ prendre le signe inférieur à l'exclusion de l'autre.

Ces racines peuvent être mises sous la forme $\lambda+\mu\sqrt{-1}$; pour cela posons $a=e^z$ et

$$U(z)+\sqrt{-1}\,V(z)=\overset{-m}{\dot{e}^z}\Big|^{+\sqrt{-1}} \qquad U(z)-\sqrt{-1}\,V(z)=\overset{-m}{\dot{e}^z}\Big|^{-\sqrt{-1}}$$

d'où

$$U(z)=\frac{\overset{-m}{\dot{e}^z}\Big|^{\sqrt{-1}}+\overset{-m}{\dot{e}^z}\Big|^{-\sqrt{-1}}}{2} \qquad V(z)=\frac{\overset{-m}{\dot{e}^z}\Big|^{\sqrt{-1}}-\overset{-m}{\dot{e}^z}\Big|^{-\sqrt{-1}}}{2\sqrt{-1}}$$

En faisant $m=-1$ ces fonctions deviennent $\cos z$ et $\sin z$. ; si l'on appelle $u(z)$ et $v(z)$ ce quelles deviennent pour m infini, la quantité $u(z) \pm v(z)\sqrt{-1}$

sera racine de l'équation $e^{ux} - x = 0$. Quand on élimine z entre les équations $u = \cos z$ $v = \sin z$ on trouve le cercle $\rho = 1$. De même si l'on élimine z entre les équations

$$\lambda = u(z) \qquad \mu = v(z)$$

on obtient la courbe

$$\text{(B)} \qquad \rho = e^{\frac{\theta}{\operatorname{tang}\theta}}$$

qui joue ainsi le même rôle que le cercle pour les fonctions circulaires.

Ajoutons que pour étudier la répartition dans le plan des racines imaginaires pour une valeur donnée de a ; les courbes definies par les équations (A) sont peu commodes ; elles ont l'une et l'autre une infinité de branches ; il vaudra mieux prendre la courbe (B) qui est invariante et la courbe $\rho = a^u$ qui n'a qu'une seule branche infinie. Mais on introduit ainsi des racines étrangères dontil ne faudra pas tenir compte.

Equation $x^x = a$. En posant $x = \dfrac{1}{y}$, $a = \dfrac{1}{\alpha}$ elle devient $y = \alpha^y$. On a donc

$$\sqrt[2]{a} = \lim\left(\begin{matrix}\pm m \\ \dot{a}^{-1}\end{matrix}\,\middle|\, e^{\pm 1}\right)^{-1} \quad \text{et} \quad \sqrt[2]{a} = \lim\left(\begin{matrix}-m \\ \dot{a}^{-1}\end{matrix}\,\middle|^{\pm\sqrt{-1}}\right)^{-1}$$

pour les racines réelles — pour les racines imaginaires

IV. Applications diverses.

Expressions des racines de quelques équations trancendantes. Intégration d'une équation aux différences mêlées. Applications diverses.

La surracine deuxième pouvant s'exprimer au moyen des algorithmes naturels, il en sera de même des fopctions qui peuvent s'y ramener.

Tel est le cas des racines de quelques équations trancendantes ; il nous suffira de ramener ces equations à l'un on à l'autre des types

$$x = a^x \qquad x^x = a$$

Equation $x^{x^m} = a$

Elevant les deux membres à la puissance m et posant $x^m = y$, $a^m = b$ elle devient
$$y^y = b$$

On a donc

$$x^m = \sqrt[\overset{2}{\cdot}]{a^m} \qquad \text{puis} \quad x = \left(\sqrt[\overset{2}{\cdot}]{a^m}\right)^{\frac{1}{m}}$$

Equation $xa^x = b$

Elevons a aux puissances exprimées par les deux membres, on a

$$\left(a^{\overset{\overset{2}{\cdot}}{x}}\right) = a^{b}$$

d'où

$$a^x = \sqrt[\overset{2}{\cdot}]{a^b} = \frac{b}{x} \qquad \text{puis} \quad x = \frac{b}{\sqrt[\overset{2}{\cdot}]{a^b}}$$

Equation $a^x = kx$.

Elle se ramène à la précédente, en l'écrivant

$$x(a^{-1})^x = k^{-1}$$

Equation $a^x = x + b$.

Elle se ramène à la précédente si on l'écrit

$$a^{x+b} = a^b(x+b)$$

et si l'on pose $\qquad x + b = y.$

On peut également réduire des équations un peu plus générales

$$(ax)^{bx^{c}} = m, \qquad (ax)^p\, b^{(ax)^{c}} = m \qquad (x+k)^m\, a^{px+q} = b$$

Pour la première posons, $ax = y$ élevons les deux membres de l'équation à la puissance $b^{-1}ca^c$ posant ensuite $y^c = z$ et $m^{b^{-1}ca^c} = n$, elle devient

$$\overset{\overset{2}{\cdot}}{z} = n$$

Pour la deuxième posons encore $ax = y$. Elevons à la puissance $p^{-1}c$ on est ramené à la forme $\qquad zA^z = B$

La troisième se ramène au même type, en posant $x + k = y$.

En résolvant ces équations, il faut d'une part ne pas tenir compte des solutions étrangères, de l'autre constater que l'on a obtenu toutes les racines de la proposée.

FIGURE 5.

Cherchons, par exemple, les intersections réelles des courbes

$$y = a^x \qquad y = bx^2$$

en supposant $a > 1$ $b > 0$. La Fig. 5 montre qu'on a toujours une racine réelle négative ; il peut aussi exister deux racines positives égales ou inégales. On a à resoudre l'équation

$$a^x = bx^2$$

on peut l'écrire $\qquad (\sqrt{a})^x = \sqrt{b}\,x$

et l'on a :

$$a = \frac{1}{\sqrt{b}} \left(\sqrt[2]{\sqrt{a}^{-\frac{1}{\sqrt{b}}}} \right)^{-1}$$

Pour que les deux racines positives soient égales, il faut avoir :

$$\sqrt[2]{\sqrt{a}^{-\frac{1}{\sqrt{b}}}} = \frac{1}{e} \quad \text{ou} \quad \sqrt{a}^{-\frac{1}{\sqrt{b}}} = \left(\frac{1}{e}\right)^{\frac{1}{e}} = \frac{1}{e^{\frac{1}{e}}}$$

C'est à dire

$$a = e^{\frac{2\sqrt{b}}{e}}$$

Si a est plus petit que cette valeur, on a deux racines réelles inégales. Il est clair qu'il faut prendre $\sqrt{a}$ et $\sqrt{b}$ avec le seul signe $+$. Pour obtenir la racine négative on changera x en $-x$, et l'on est ramené à l'équation $bx^2 = \frac{1}{a^x}$ qu'on peut encore écrire

$$x\sqrt{a}^x = \frac{1}{\sqrt{b}} \quad \text{et l'on a}$$

$$x = \frac{1}{\sqrt{b}} \left(\sqrt[2]{\sqrt{a}^{\frac{-1}{\sqrt{b}}}} \right)^{-1}$$

qu'il faudra changer de signe. Ainsi les 3 racines réelles possibles sont données par la même formule, $\sqrt{a}$ étant toujours positif, et $\sqrt{b}$ étant postitif pour les deux racines positives, et negatif pour la racine négative.

Fonction $\quad y = \sqrt{a^{2x} - x^2}$.

Nous avons rencontré cette fonction en étudiant la répartition des racines imaginaires de l'équation

$$z = a^z.$$

En coordonnées polaires l'équation de cette courbe est $\rho = a^{\rho\cos\omega}$. En coordonnées semi-polaires, abscisse et rayon vecteur, elle est $\rho = a^x$. Pour $a > 1$, elle peut présenter trois formes suivant que, a est plus petit que $e^{\frac{1}{e}}$, égal à $e^{\frac{1}{e}}$ ou plus grand que $e^{\frac{1}{e}}$. On trouve facilement que pour x infini et positif elle est asymptote à

$y = \pm a^{x}$ qu'elle est tangente à ces courbes pour $x = o$; qu'en chacun des points réels où elle coupe ox, elle a une tangent eparallèle à oy. Cherchons les abscisses de ces points d'intersection.

FIGURES 6, 7, 8.

Pour $y = o$ il faut avoir $a^{x} = x$ ou $a^{-x} = x$; les deux racines réelles positives sont les deux valeurs réelles de $(\sqrt[2]{a})^{-1}$ égales si $a = e^{\frac{1}{e}}$, inégales si $a < e^{\frac{1}{e}}$ la racine négative est $(\sqrt[2]{a^{-1}})^{-1}$. Cherchons les abscisses des points pour lesquels la courbe a des tangentes horizontales. On a :

$$y' = \pm \frac{a^{2x} \mathrm{L}a - x}{\sqrt{a^{2x} - x^2}}$$

les racines du numérateur s'obtiendront en posant $x = \frac{\mathrm{L}a}{z}$ il vient

$$a^{2\mathrm{L}a} = \frac{1}{z^z}$$

par suite

$$z = \sqrt[2]{a^{-2\mathrm{L}a}} \cdot \qquad \text{et} \qquad x = \mathrm{L}a(\sqrt[2]{a^{-2\mathrm{L}a}})^{-1}$$

les deux valeurs seront distinctes si l'on a $1 < a^{2\mathrm{L}a} < e^{\frac{1}{e}}$; elles seront égales et on aura un point stationnaire si l'on a

$$a^{2\mathrm{L}a} = e^{\frac{1}{e}} \qquad \text{ou} \qquad 2(\mathrm{L}a)^2 = \frac{1}{e} \qquad \text{c'est à dire} \quad \mathrm{L}a = \frac{1}{\sqrt{2e}}$$

l'abscisse et l'ordonnée ont alors la valeur commune $\sqrt{\frac{e}{2}}$

Autre appplication.

Comme exemple d'équations qu'on intégre en les différentiant M. H. Laurent *

cite l'équation (1) $x = e^{y'} + y'$

En différentiant on a : $dx = e^{y'} dy' + dy'$

en multipliant par y' $dy = y'(e^{y'} + 1)dy'$

d'où $y = \int y'(e^{y'} + 1)dy' + \mathrm{C}^{te}$

C'est a dire (2) $y = e^{y'}(y' - 1) + \frac{1}{2}q'^2 + \mathrm{C}^{te}$

* *Traité d'analyse.* Tome V.

On peut éliminer entre (1) et (2) y' et il reste une relation implicite entre x et y. L'algorithme de la surpuissance permet d'expliciter y ; en effet (1) peut s'écrire

$$(x - y') = e^{y'} \qquad \text{ou} \qquad (x - y')e^{x - y'} = e^{x}$$

Elevons e aux puissances indiquées par les deux membres ; on a

$$\left(e^{\overset{2}{\dot{x}} - y'}\right) = \overset{2}{\dot{e}} \,\Big|\, x$$

On a donc

$$e^{x - y'} = \sqrt[\overset{2}{\cdot}]{\overset{2}{\dot{e}} \,\Big|\, x} \qquad e^{y'} = e^{x}\left[\sqrt[\overset{2}{\cdot}]{\overset{2}{\dot{e}} \,\Big|\, x}\right]^{-1} \qquad y' = x - e^{x}\left[\sqrt[\overset{2}{\cdot}]{\overset{2}{\dot{e}} \,\Big|\, x}\right]^{-1}$$

Remplaçant y' et $e^{y'}$ par leurs valeurs dans (2), on aura explicité y.

Intégration de l'équation aux différences mêlées

$$y^{(m+p)} + \Delta y^{(m+p)} = a y^{(m)}$$

On peut l'écrire $\qquad y_1^{(m+p)} - a y^{(m)} = 0$

ou encore $\qquad y^{(m+p)} - a y_{-1}^{(m)} = 0$

y_1 y_{-1} désignant la valeur de la fonction quand x est changé en $x + 1$, ou en $x - 1$ comme l'on a

$$y_{-1}^{(m)} = y^{(m)} - \frac{1}{1!} y^{(m+1)} + \frac{1}{2!} y^{(m+2)} - \ . \ . \ .$$

On a à résoudre l'équation linéaire à coefficients constants

$$o = y^{(m+p)} - a\left(y^{(m)} - \frac{1}{1!} y^{(m+1)} + \ . \ . \ . \ \right)$$

L'équation caractéristique peut s'écrire

$$r^{m+p} - a r^{m} e^{-r} = o$$

Elle admet les m recines $r = o$; ce qui donne un polynôme de degré m, il reste $r^{p} = a e^{-r}$.

Pour la résoudre élevons a la puissance $\dfrac{1}{p}$

$$r a^{-p^{-1}} = \left(e^{-p^{-1}}\right)^{r}$$

ce qui la ramène à la forme $Ar = B^r$ et les racines seront

$$r = \pm a^{p^{-1}} \left(\sqrt[2]{e^{p^{-1} a^{p^{-1}}}} \right) - 1$$

Suivant les valeurs attribuées à a et à p il pourra exister une ou trois racines réelles P, R, S* que fournissent trois intégrales particulières

$$y = C_{m+1} e^{Px} \qquad y = C_{m+2} e^{Rx} \qquad y = C_{m+3} e^{Sx}$$

Pour les racines imaginaires que sont conjuguées deux à deux et d'ordre simple, on aura

$$r = a^{p^{-1}} \left\{ u\left(-p^{-1} a^{p^{-1}} \right) + \sqrt{-1}\, v\left(-p^{-1} a^{p^{-1}} \right) \right\}$$

u et v étant les fonctions définies au paragraphe III. En ne tenant pas compte des racines étrangères que l'on reconnaitra facilement il restera une infinité d'intégrales particulières de la forme

$$y = e^{a^{p^{-1}} u(z) x} \left\{ A \cos\left(a^{p^{-1}} v(z) x \right) + B \sin\left(a^{p^{-1}} v(z) x \right) \right\}$$

où l'on a posé $-p^{-1} a^{p^{-1}} = z$

et où A et B sont des constantes arbitraires.

En ce qui concerne l'expression des hyperlogarithmes comme limites d'expressions directes propres à représenter leur calcul, et le développement de la fonction surexponentielle en série quand l'exposant n'est pas un nombre entier je suis arrivé à des résultats qui ne pourraient trouver place ici sans allonger de beaucoup cette note ; j'en ferai l'objet d'une communication ultérieure.

Notes bibliographiques sur la convergence des substitutions uniformes

Koenigs : Sur certaines équations fonctionnelles (*Annales de l'Ecole. Normale*, 1887.)

Korkine : Sur un problème d'interpolation (*Bulletin des Sc. Math.*, 1882.)

Je n'ai trouvé aucun renseignement bibliographique sur le quatrième algorithme naturel.

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* Deux racines pourraint être égales.

Un correspondant de St Pétersbourg m'a appris qu'Euler avait donné les racines réelles, mais non les racines imaginaires de l'équation $x = a^x$ par des expressions qui ont le même sens que celles que je donne ici) son mémoire a paru dans les "Acta Ac. Sc. Petropolitanae pro anno MDCCLXXVII.; je n'ai pu encore me procurer ce recueil.

Une partie des résultats consignés dans cette note a èté publiée dans :

Association française pour l'Avancement des Sciences, Bordeaux, 1895.

Nouvelles Annales de Mathématiques. Laisant et Antomari, Décembre 1896, et Février 1897.

Second Meeting, December 10th, 1897.

J. B. Clark, Esq., M.A., F.R.S.E., President, in the Chair.

On some questions in Arithmetic.

By Prof. Steggall.

Note on the Transformations of the Equations of Hydrodynamics.

By H. S. Carslaw, M.A., Glasgow and Cambridge.

This summer there came into my hands a copy of the spring issue of the *Mittheilungen der Math. Gesellschaft in Hamburg* containing a paper on the "Transformationen der hydrodynamischen Gleichungen mit Berücksichtigung der Reibung."

On examination, I found embodied in the somewhat lengthy communication practically the following method, which I had entered in my notes three years ago when working at the subject. Thinking that it was bound to have been used earlier, I simply preserved it, as likely to prove useful if I were ever called upon to teach Hydrodynamics.

The fact of the aforesaid paper being afforded a prominent place in that German journal prompts me to submit this note to the Society. If the method is new, its publication in English seems not uncalled for.

§1.

The equations of motion in a viscous liquid, with regard to fixed rectangular axes, are accepted as

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \mathrm{X} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u,$$

and two others,

where $\dfrac{\mathrm{D}f}{\mathrm{D}t}$ stands for $\left(\dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} + v\dfrac{\partial}{\partial y} + w\dfrac{\partial}{\partial z}\right)f.$

Now in this case $\nabla^2 u = 2\left\{\dfrac{\partial\eta}{\partial z} - \dfrac{\partial\zeta}{\partial y}\right\}$

ξ, η, ζ being the components of molecular rotation at the point (x, y, z).

Thus we have our equations in the form,

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \mathrm{X} - \frac{1}{\rho}\frac{\partial p}{\partial x} + 2\nu\left\{\frac{\partial \eta}{\partial z} - \frac{\partial \zeta}{\partial y}\right\},$$

in which, since (ξ, η, ζ) are the components of a vector, we are able easily to transform to any system of orthogonal axes, and as special cases we have the spherical polar and the cylindrical systems.

§ 2.

Taking the system defined by

$$ds^2 = h_1^2 d\alpha^2 + h_2^2 d\beta^2 + h_3^2 d\gamma^2,$$

let the axes of (x, y, z) coincide with those of (α, β, γ) for the point (α, β, γ).

Then we have merely to consider the alteration in our equations when the (u, v, w) at $(\alpha + d\alpha, \beta + d\beta, \gamma + d\gamma)$ are measured along the (α, β, γ) axes there.

If we are dealing with vectors this only involves the determination of the infinitesimal rotations of the axes.

As usual, we have

$$d\theta_3 = \frac{1}{h_1}\frac{\partial h_2}{\partial \alpha}\,.\,d\beta - \frac{1}{h_2}\frac{\partial h_1}{\partial \beta}\,.\,d\alpha,$$

and two other equations.

Then for any vector with components U, V, W, along the axes,

$$\delta \mathrm{U} = d\mathrm{U} - \mathrm{V}d\theta_3 + \mathrm{W}d\theta_2,\text{*}$$

giving

$$\left(\frac{\partial \mathrm{U}}{\partial x}\right) = \frac{1}{h_1}\frac{\partial \mathrm{U}}{\partial \alpha} + \frac{\mathrm{V}}{h_1 h_2}\frac{\partial h_1}{\partial \beta} + \frac{\mathrm{W}}{h_1 h_2}\frac{\partial h_1}{\partial \gamma},$$

$$\left(\frac{\partial \mathrm{U}}{dy}\right) = \frac{1}{h_2}\frac{\partial \mathrm{U}}{\partial \beta} - \frac{\mathrm{V}}{h_1 h_2}\frac{\partial h_2}{\partial \alpha},$$

$$\left(\frac{\partial \mathrm{U}}{\partial z}\right) = \frac{1}{h_3}\frac{\partial \mathrm{U}}{\partial \gamma} - \frac{\mathrm{W}}{h_1 h_3}\frac{\partial h_3}{\partial \alpha}.$$

§ 3.

We have now to apply this to the case in question.

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* For this and the first results in the next paragraph, see § 121 of *Love's Elasticity*, Vol. I.

Considering (u, v, w),

$$2\zeta = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial \alpha}\cdot(h_2 v) - \frac{\partial}{\partial \beta}(h_1 u)\right\}.$$

Likewise from (ξ, η, ζ)

$$\frac{\partial \eta}{\partial z} - \frac{\partial \zeta}{\partial y} = \frac{1}{h_2 h_3}\left\{\frac{\partial}{\partial \gamma}(\eta h_2) - \frac{\partial}{\partial \beta}(\zeta h_3)\right\}$$

Remembering the meaning of $\frac{\mathrm{D}u}{\mathrm{D}t}$, we find

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + \frac{1}{h_1} u \frac{\partial u}{\partial \alpha} + \frac{1}{h_2} v \frac{\partial u}{\partial \beta} + \frac{1}{h_3} w \frac{\partial u}{\partial \gamma}$$

$$- \frac{v}{h_1 h_2}\left\{v \frac{\partial h_2}{\partial \alpha} - u \frac{\partial h_1}{\partial \beta}\right\} + \frac{w}{h_1 h_3}\left\{u \frac{\partial h_1}{\partial \gamma} - w \frac{\partial h_3}{\partial \alpha}\right\},$$

and our first equation of motion becomes

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \mathrm{X} - \frac{1}{\rho h_1}\frac{\partial p}{\partial \alpha} + \frac{2\nu}{h_2 h_3}\left\{\frac{\partial}{\partial \gamma}(\eta h_2) - \frac{\partial}{\partial \beta}(\zeta h_3)\right\}.$$

The other two follow in the same way.

§ 4.

The advantage of this method seems to consist in its straightforwardness and also in the fact that we are almost bound to have had already before us the expression for (ξ, η, ζ) in the general coordinates. In the spherical polar coordinates they are very easily calculated, and the transformation to the (r, θ, ϕ) system is thus a simple one.

To deduce the equations for that case we have to take

$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = r\sin\theta.$$

This gives

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial u}{\partial \phi} - \frac{v^2 + w^2}{r}$$

$$= \mathrm{R} - \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{2\nu}{r^2\sin\theta}\left\{\frac{\partial}{\partial \phi}(\eta r) - \frac{\partial}{\partial \theta}(\zeta r\sin\theta)\right\}$$

$$= \mathrm{R} - \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{2\nu}{r\sin\theta}\left\{\frac{\partial \eta}{\partial \phi} - \frac{\partial}{\partial \theta}(\zeta\sin\theta)\right\}$$

Similar equations result for the other two directions, and these can be reduced to depend only on (u, v, w) by substituting for (ξ, η, ζ) their values as given above.

A New Proof of the Formulae for Right-Angled Spherical Triangles.

By Professor JOHN JACK.

It is assumed that the sines of the sides are proportional to the sines of the opposite angles.

ACB (Fig. 9) is a spherical $\triangle$, with C a right angle.

Produce AC, AB to D and E so that $AD = AE = \frac{\pi}{2}$. Draw the great $\odot$ DEF and produce CB to meet it in F.

Then F is the pole of AD and A the pole of DF.

Then sides and angles of ABC are $\quad a \quad b \quad c \quad A \quad B$

BEF are $\quad \frac{\pi}{2} - A \quad \frac{\pi}{2} - c \quad \frac{\pi}{2} - a \quad B \quad \frac{\pi}{2} - b.$

$$\therefore \quad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{1} \quad - \quad - \quad - \quad \text{I.}$$

and
$$\frac{\sin(\frac{\pi}{2} - A)}{\sin B} = \frac{\sin(\frac{\pi}{2} - c)}{\sin(\frac{\pi}{2} - b)} = \frac{\sin(\frac{\pi}{2} - a)}{1}$$

that is
$$\frac{\cos A}{\sin B} = \frac{\cos c}{\cos b} = \frac{\cos a}{1} \quad - \quad - \quad - \quad \text{II.}$$

and $\therefore$
$$\frac{\cos B}{\sin A} = \frac{\cos c}{\cos a} = \frac{\cos b}{1} \quad - \quad - \quad - \quad \text{III.}$$

by interchange of a, A and b, B.

$$\therefore \quad \left.\begin{aligned} \sin a &= \sin c \sin A \\ \sin b &= \sin c \sin B \end{aligned}\right\} \text{ from I.} \quad - \quad - \quad - \quad 1.$$

$$\text{and} \quad \left.\begin{aligned} \cos A &= \cos a \sin B \\ \cos B &= \cos b \sin A \end{aligned}\right\} \text{ from II., III.} \quad - \quad - \quad 2.$$
and

and $\quad \cos c = \cos a . \cos b \quad$ from II. or III. $\quad - \quad 3.$

From 2 $\quad \cos A \cos B = \cos a \cos b \sin A \sin B$

$$\therefore \quad \cot A \cot B = \cos a \cos b = \cos c \text{ by } 3 \quad - \quad - \quad - \quad 4.$$

Again
$$\sin c = \frac{\sin b}{\sin B} \quad \text{by I.}$$

$$\cos c = \frac{\cos A \cos b}{\sin B} \quad \text{by II.} \qquad \text{Divide}$$

$$\therefore \quad \tan c = \frac{\tan b}{\cos A} \qquad \therefore \quad \left.\begin{aligned} \tan b &= \tan c \cos A \\ \text{and } \tan a &= \tan c \cos B \end{aligned}\right\} \quad - \quad - \quad 5$$

Again $\qquad \sin a = \dfrac{\sin b \sin A}{\sin B}$ by I.

and $\qquad \cos a = \dfrac{\cos A}{\sin B}$ by II. Divide

$$\left.\begin{array}{ll} \therefore & \tan a = \tan A \sin b \\ \text{so} & \tan b = \tan B \sin a \end{array}\right\} \quad - \quad - \quad - \quad - \quad 6.$$

Note on Napier's Rules.

By Professor John Jack.

Denote the parts $\quad b \;\; A \;\; c \;\; B \;\; a \quad$ of $\triangle$ ABC (Fig. 9)

by $\quad 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5$

then the parts corresponding of the $\triangle$ BEF, namely,

$$\frac{\pi}{2} - c, \; B, \; \frac{\pi}{2} - a, \; \frac{\pi}{2} - b, \; \frac{\pi}{2} - A$$

will be denoted by $\frac{\pi}{2} - 3, \; 4, \; \frac{\pi}{2} - 5, \; \frac{\pi}{2} - 1, \; \frac{\pi}{2} - 2$.

Now a third $\triangle$ can similarly be derived from this second, a fourth from the third, and a fifth from the fourth. But when the process is applied to the fifth, the first $\triangle$ is obtained. Hence only 5 $\triangle$s can be obtained, which are the following:—

1	2	3	4	5
$\frac{\pi}{2} - 3$	4	$\frac{\pi}{2} - 5$	$\frac{\pi}{2} - 1$	$\frac{\pi}{2} - 2$
5	$\frac{\pi}{2} - 1$	2	3	$\frac{\pi}{2} - 4$
$\frac{\pi}{2} - 2$	3	4	$\frac{\pi}{2} - 5$	1
$\frac{\pi}{2} - 4$	$\frac{\pi}{2} - 5$	$\frac{\pi}{2} - 1$	2	$\frac{\pi}{2} - 3$
1	2	3	4	5

where the mid-column contains the hypotenuse, the two next to it contain the angles, and the extreme columns the sides of the several right-angled triangles.

Now *assume* cos hypotenuse = product of cosines of sides
= product of cotangents of angles ;
that is sine complement of hypotenuse
= product of cosines of sides
= product of tangents of complements of angles.

Now change the 2nd, 3rd, and 4th columns to complements and we have

1	$\frac{\pi}{2}-2$	$\frac{\pi}{2}-3$	$\frac{\pi}{2}-4$	5
$\frac{\pi}{2}-3$	$\frac{\pi}{2}-4$	5	1	$\frac{\pi}{2}-2$
5	1	$\frac{\pi}{2}-2$	$\frac{\pi}{2}-3$	$\frac{\pi}{2}-4$
$\frac{\pi}{2}-2$	$\frac{\pi}{2}-3$	$\frac{\pi}{2}-4$	5	1
$\frac{\pi}{2}-4$	5	1	$\frac{\pi}{2}-2$	$\frac{\pi}{2}-3$

where, taking any horizontal line,
sine of mid column = product of tangents of adjoining columns
= product of cosines of extreme columns ;
and this proves completely Napier's rules, for each horizontal line contains Napier's parts in the same (cyclic) order.

Third Meeting, 14th January 1898.

J. B. CLARK, Esq., M.A., F.R.S.E., President, in the Chair.

The Trisection of a Given Angle.

By LAWRENCE CRAWFORD, M.A., B.Sc.

1. In a given circle let the arc AP subtend an angle $3a$ at the centre O, it is required to trisect the angle AOP, or the arc AP.

The three trisectors will be OQ_1, OQ_2, OQ_3, where $AOQ_1 = a$, $AOQ_2 = \frac{1}{3}(2\pi + 3a) = a + 2\pi/3$, $AOQ_3 = \frac{1}{3}(4\pi + 3a) = a + 4\pi/3$

$\therefore$ Q_1, Q_2, Q_3 form an equilateral triangle. (See Figs. 10 and 11.)

We proceed to solve the problem by drawing a conic through Q_1, Q_2, Q_3, and we wish to find in what cases such a conic can be drawn, a conic cutting the circle in four points, three of which form an equilateral triangle.

2. Take the circle as $x^2 + y^2 = d^2$, OA being the axis of x, and let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

On eliminating y, the equation for the x co-ordinate of a point of intersection is found to be

$$x^4\{(a-b)^2 + 4h^2\} + 4x^3\{g(a-b) + 2fh\} + 2x^2\{(a-b)(bd^2 + c) + 2g^2 + 2f^2 - 2h^2d^2\} + 4x\{g(bd^2 + c) - 2fhd^2\} + (bd^2 + c)^2 - 4d^2f^2 = 0,$$

or, for brevity, $Ax^4 + 4Bx^3 + 2Cx^2 + 4Dx + E = 0.$

We can write down the sum of the four roots, the sum of their products two by two, the sum of their products three by three, and the product of the four in terms of these coefficients; but three of the roots, x_1, x_2, x_3, say, are $d\cos a$, $d\cos(a + 2\pi/3)$, $d\cos(a + 4\pi/3)$ $\therefore$ their sum is zero, the sum of their products two by two is $-3d^2/4$, and their product $1/4\, d^3\cos 3a$, and the substitution of these values gives us, if x_4 be the fourth root,

$$Ax_4 = -4B, \qquad -A \, . \, 3d^2/4 = 2C,$$

$$Ad^3\cos 3a/4 - Ax_4 \, . \, 3d^2/4 = -4D, \qquad Ad^3\cos 3a \, . \, x_4/4 = E,$$

which, on eliminating x_4 give

$$3Ad^2 + 8C = 0, \qquad Ad^3\cos 3a + 12d^2B + 16D = 0,$$

and
$$Bd^3\cos 3a + E = 0. \qquad \text{(I.)}$$

Now if we had started by eliminating x and obtaining a biquadratic in y, a similar procedure, on noting that three of the roots are $d\sin\alpha$, $d\sin(\alpha+2\pi/3)$, and $d\sin(\alpha+4\pi/3)$, would have given us the equations

$$3\text{A}d^2+8\text{C}'=0, \qquad -\text{A}d^3\sin 3\alpha+12d^2\text{B}'+16\text{D}'=0,$$

and

$$-\text{B}'d^3\sin 3\alpha+\text{E}'=0, \qquad \text{(II.)}$$

where B′, C′, D′, E′ are coefficients corresponding to B, C, D, E; the coefficient of the leading term in both biquadratics is the same.

3. These equations, (I.) and (II.), are the equations to be satisfied, we proceed to reduce them and to find which are independent.

From the first equation of each set we see that C, C′ are equal, *i.e.*,

$$(a-b)(bd^2+c)+2f^2+2g^2-2h^2d^2=-(a-b)(ad^2+c)+2f^2+2g^2-2h^2d^2$$

$$\therefore \quad (a-b)(\overline{a+b}d^2+2c)=0.$$

Now $(a-b)$ cannot be zero, as our conditions (I.) and (II.) hold wherever we take the axes, keeping the origin at O, but $(a-b)$ is not an invariant for such change of axes $\therefore$ cannot always be zero, $\therefore$ we have the condition

$$(a+b)d^2+2c=0. \qquad (1)$$

With the help of this equation to eliminate C, the first equation of either (I.) or (II.) gives on simplification

$$d^2\{(a-b)^2+4h^2\}=16(f^2+g^2). \qquad (2)$$

The other two equations of (I.) give now on reduction, using (1) and (2) where necessary,

$$\{(a-b)^2+4h^2\}d\cos 3\alpha=4\{2fh-g(a-b)\} \qquad (3)$$

and

$$\{2fh+g(a-b)\}d\cos 3\alpha=h^2d^2-4g^2,$$

but on multiplying crosswise, we find these are not independent, so we may omit the latter.

The simplification of the last two equations of (II.) gives two equations of a similar type, also not independent; the former of these is

$$-\{(a-b)^2+4h^2\}d\sin 3\alpha=4\{2gh+f(a-b)\}. \qquad (4)$$

Now if we square equations (3) and (4) and add, we find after division by $(a-b)^2+4h^2$ that we only get equation (2), hence equations (1), (3), and (4) are our only independent equations.

From the last two we can solve for f and g in terms of a, h, and b, and using equation (1) for c, we have that the equation of the conic may be written

$$2ax^2 + 4hxy + 2by^2 - dx\{2h\sin 3a + (a-b)\cos 3a\} - dy\{-2h\cos 3a + (a-b)\sin 3a\} - (a+b)d^2 = 0.$$

4. From the equation $Ax_4 = -4B$, and the corresponding one, we get the coordinates x_4, y_4 of the fourth point of intersection,

$$x_4\{(a-b)^2+4h^2\} = d\cos 3a\{(a-b)^2-4h^2\} + 4(a-b)hd\sin 3a,$$

$$y_4\{(a-b)^2-4h^2\} = -d\sin 3a\{(a-b)^2-4h^2\} + 4(a-b)hd\cos 3a.$$

Note that for a conic with axes parallel to the coordinate axes, $h=0$ and the fourth point of intersection is $(d\cos 3a,\ -d\sin 3a)$, a fixed point for all such conics.

5. Take now some special examples. Suppose first the conic is a rectangular hyperbola, then $a+b=0$, $\therefore$ equation (1) gives that c is zero, or that the conic goes through the origin: this, however, we know, for the origin is the orthocentre of the equilateral triangle $Q_1Q_2Q_3$ and a rectangular hyperbola circumscribing a triangle passes through its orthocentre.

The equation of the rectangular hyperbola with axes parallel to the coordinate axes reduces to

$$x^2 - y^2 - dx\cos 3a - dy\sin 3a = 0,$$

so only one rectangular hyperbola with such axes can be drawn to solve our problem.

Its centre is $(\frac{1}{2}d\cos 3a,\ -\frac{1}{2}d\sin 3a)$, and its transverse axis is $d\sqrt{\cos 6a}$ parallel to the axis of x, if $\cos 6a$ be positive, but $d\sqrt{-\cos 6a}$ parallel to the axis of y, if $\cos 6a$ be negative.

Another simple solution can be got by making the rectangular hyperbola pass through A instead of putting h zero.

The equation of the hyperbola is

$$a(x^2-y^2) + 2hxy - dx(h\sin 3a + a\cos 3a) + dy(h\cos 3a - a\sin 3a) = 0$$

$\therefore$ if it pass through the point A, $(d,\ o)$ we find

$$h = a\tan\frac{3a}{2},$$

and the equation reduces to

$$x_4^2 - y^2 + 2\lambda xy - dx - d\lambda y = 0,$$

where λ is $\tan\frac{3a}{2}$, and this is a unique conic.

The centre of this conic is the point $(\frac{1}{2}d,\ o)$ $\therefore$ the radius d of the circle is a diameter of the conic. Figure 10 is a rough drawing of the circle and the conic, A being here the fourth point of intersection. This is the construction given in Taylor's "Geometry of Conics," Example 528.

6. Take now the conic as a hyperbola, eccentricity 2, the condition for which is $(3a+b)(a+3b)=4\,h^2$.

Draw a conic with axes parallel to the axes of coordinates, then h is zero, take here $3a+b$ zero, then the equation of the conic reduces to

$$x^2-3y^2-2dx\cos3a-2dy\sin3a+d^2=0,\quad \text{a unique conic.}$$

The centre of this conic is $(d\cos3a,\ -\frac{1}{3}d\sin3a)$, and transferred to parallel axes through this point, the equation of the conic is $x^2-3y^2+\frac{4}{3}d^2\sin3a=0$, so that the transverse axis is parallel to the axis of y and is in length $^4/_3 d\sin3a$.

The fourth point of intersection is $(d\cos3a,\ -d\sin3a)$, so that the distance between it and the centre is the semi-transverse axis, hence it is a vertex.

Also, if C be the centre of the conic, CP is equal to the transverse axis, *i.e.*, twice the semi-tranverse axis, but the eccentricity is 2, so that P is a focus. The other vertex is the point $(d\cos a,\ ^1/_3 d\sin3a)$.

A rough figure is drawn (Fig. 11) to show this construction, which may be put as follows: Take Q_4 the end of the ordinate through P and trisect it in V and C. With C as centre, V and Q_4 as vertices, describe a hyperbola of eccentricity 2, this will cut the circle in the points of trisection of the angle AOP. This is the construction given in Taylor's Conics, Example 390.

7. The conic which solves the problem may be a parabola or an ellipse, but these conics do not give such simple constructions as those two which have been drawn.

The Centre of Gravity of a Circular Arc.

By Mr G. E. CRAWFORD, M.A.

To find the Centre of Gravity of a Circular Arc.

Let a (Fig. 12) be the radius, 2α the angle at the centre, AB the arc, of total mass m, G its centre of gravity symmetrically situated.

Imagine the arc to be part of a *circle of string* rotating uniformly with velocity u round C and of linear density ρ

Then if T be the Tension at either extremity

$$\text{Resolving} \qquad 2T\sin\alpha = \text{Force to centre}$$

$$= mdw^2 = 2\rho a\alpha dw^2$$

$$\therefore \quad T \quad = \quad a\rho w^2 \frac{ad}{\sin\alpha}$$

But T being constant, this formula must be constant, and $\therefore$ true for all values of α

$$\therefore \quad \frac{ad}{\sin\alpha} \quad \text{is constant.}$$

But when $\alpha = 0$ its value is a

$$\therefore \quad d = a\frac{\sin\alpha}{\alpha}.$$

A Demonstration of the Apparatus used in Practical Skiagraphy by the Röntgen Rays.

By Dr HARRY RAINY.

Fourth Meeting, 11th February 1898.

J. B. CLARK, Esq., M.A., F.R.S.E., President, in the Chair.

On a Geometrical Problem.

By R. GUIMARÃES.

Stewart, a Scotch geometrician, gave in 1763 the demonstration of the following theorem: "If we divide the base of a triangle into two segments by a straight line going through the vertex, the sum of the squares of the two sides, multiplied each by the non-adjacent segment, is equal to the product of the base multiplied by the square of the straight line plus the rectangle contained by the two segments."

Recently, the Belgian geometer, Clément Thiry,* presented under various forms the formula which represents Stewart's theorem, drawing from it numerous classical propositions. As for us, we have used this theorem to resolve many geometrical problems.†

At present we wish to show that Stewart's theorem enables us to resolve easily the following problem: "To draw a circle touching another given circle and passing through two given points."

The enunciation of Stewart's theorem ‡ most useful for our purpose is as follows: Three points A, B, C in a straight line being given, the distances between them and any given point O are satisfied by

$$OA^2 . BC + OB^2 . CA + OC^2 . AB + AB . BC . CA = 0 \quad - \quad (1)$$

where we must consider Euler's identity

$$AB + BC + CA = 0.$$

* Sur le théorème de Stewart (*Revue de l'Instruction Publique*), Bruxelles, 1887; applications remarquables du théorème de Stewart et théorie du barycentre, Bruxelles, 1891.

† *El Progreso Matemático*, Zaragoza, 1892, pp. 62, 94, 124.

‡ Clément Thiry, *Op. Cit.*, p. 6.

Now the potencies a, b, c of the points A, B, C relatively to the circle O are, R being the radius of the circle,

$$a = OA^2 - R^2, \qquad b = OB^2 - R^2 \qquad c = OC^2 - R^2$$

which transform equation (1) into

$$a \, . \, BC + b \, . \, CA + c \, . \, AB + AB \, . \, BC \, . \, CA = 0 \quad - \quad - \quad (2)$$

But CB . CA = c, if C be the point where the tangent common to the two circles meets AB.

Thus equation (2) becomes

$$a \, . \, BC + b \, . \, CA = 0$$

or

$$\frac{CA}{CB} = \frac{a}{b} \quad - \quad - \quad - \quad - \quad - \quad (3)$$

Thus the point C is determined and the problem solved. If the problem is possible, A and B must both be inside or both outside the circle O.

Mr Muirhead suggests the following Solution.

A and B are the given points, DEF the given ⊙, and ABD the required ⊙

In virtue of the equality of angles indicated in Fig. 15, we have

$$\frac{\text{Power of A}}{\text{Power of B}} = \frac{AD \, . \, AF}{BD \, . \, BE}$$

$$= \frac{AD^2}{BD^2} = \frac{CA}{CB}$$

Discussion on Euclid's Definition of Proportion.

Papers by Prof. GIBSON and Mr W. J. MACDONALD.

Fifth Meeting, March 11th, 1898.

Dr MORGAN, Vice-President, in the Chair.

An Analysis of all the Inconclusive Votes possible with 15 Electors and 3 Candidates.

By Professor STEGGALL.

A Suggestion for a Shortened Table of Five-Figure Logarithms.

By Professor STEGGALL.

Note on the Centre of Gravity of a Circular Arc.

By JOHN DOUGALL, M.A.

Mr Crawford's note on this subject, read at a recent meeting, reminds me of a method I gave to a class four or five years ago.

FIGURE 14.

Let AMB be an arc subtending an angle 2α at the centre O of a circle of radius a. The centre of gravity G_1 lies, from symmetry, on OM the line from O to the mid-point of the arc.

Let G_2 be the C.G. of an adjacent arc BNC of angle 2β.

If G be the C.G. of the whole arc AMBNC, the angle AOG is $\alpha + \beta$.

Thus $\angle G_1OG = \beta$ and $\angle G_2OG = \alpha$.
Also G_1GG_2 is a straight line.

But $GG_1 : GG_2 =$ mass at G_2 : mass at G_1

$$= \beta : \alpha$$

and $\quad GG_1 : GG_2 = OG_1 \sin\beta : OG_2 \sin\alpha$

$\therefore\ OG_1 \cdot \dfrac{\alpha}{\sin\alpha} = OG_2 \cdot \dfrac{\beta}{\sin\beta}$, and therefore each must be a constant.

By taking the arc indefinitely small, we get the constant equal to a the radius, and therefore $OG_1 = \dfrac{a \sin\alpha}{\alpha}$.

It is curious to observe that the result may be deduced, though not quite so simply, from the mere consideration that G is in the line G_1G_2.

Thus $$\triangle G_1OG_2 = \triangle G_1OG + \triangle GOG_2$$

giving $$\frac{\sin(\alpha+\beta)}{OG} = \frac{\sin\alpha}{OG_1} + \frac{\sin\beta}{OG_2};$$

or, if we denote the function of α, $\dfrac{\sin\alpha}{OG_1}$, by $\phi(\alpha)$,

$$\phi(\alpha+\beta) = \phi(\alpha) + \phi(\beta)$$

and $\therefore\ \phi(\alpha) =$ a constant multiple of α, as before.

Extension of the Notion of Wave-surface to Space of n Dimensions.

By Professor P. H. SCHOUTE.

1. The following two modes of generation of the wave-surface are pretty generally known.

(*a*) A given ellipsoid $E \equiv x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 - 1 = 0$ (surface of elasticity) is cut by any central plane π along an ellipse of semi-axes λ_1 and λ_2. If π varies, the two pairs of planes π' parallel to π at distances k^2/λ_1, k^2/λ_2 (k = constant) from it envelope the wave-surface W_1 represented by the tangential equation

$$\sum_{i=1}^{i=3} a_i^2 u_i^2/(a_i^2 - k^4 \Sigma u_i^2) = 0,$$

if the tangential coordinates u_i depend on $\Sigma ux + 1 = 0$.

(*b*) A given ellipsoid $V \equiv a_1^2 x_1^2 + a_2^2 x_2^2 + a_3^2 x_3^2 - k^4 = 0$ (surface of velocity) is cut by any central plane π along an ellipse of semi-axes μ_1, μ_2. If π varies, the locus of the two pairs of points P′, situated in the normal to π through the centre O of V and at distances μ_1, μ_2 from 0, is the wave-surface W_2 represented by the equation

$$\sum_{i=1}^{i=3} x_i^2/(k^4 - a_i^2 \Sigma x_i^2) = 0.$$

The surfaces W_1 and W_2 coincide. This result found by Professor Mannheim, who gave a very elegant geometrical demonstration of it (*Annuaire de l'association française, Congrès de Lille*, 1874) is not easily verified by analysis.

2. In the *Nieuw Archief* (series 2, vol. 3, p. 239, 1897) I have extended the two modes of generation stated above to n-dimensional space S^n.

My results are as follows:—

(*a*) A given quadratic space of $n-1$ dimensions represented by $E^{n-1} \equiv \sum_{i=1}^{i=n} x_i^2/a_i^2 - 1 = 0$ is cut by any central linear space of $n-1$ dimensions σ in a quadratic space E^{n-2} of semi-axes $\lambda_1 \ \lambda_2 \ldots . \lambda_{n-1}$.

If σ varies, the $n-1$ pairs of linear spaces σ' parallel to σ at distances $k^2/\lambda_1,\ k^2/\lambda_2,\ \ldots\ k^2/\lambda_{n-1}$ from it envelope the wave-space W_1^{n-1} represented by

$$\sum_{i=1}^{i=n} a_i^2 u_i^2/(a_i^2 - k^4 \Sigma u_i^2) = 0.$$

(β) A given quadratic space represented by

$$V^{n-1} \equiv \sum_{i=1}^{i=n} a_i^2 x_i^2 - k^4 = 0$$

is cut by any central linear space σ in a quadratic space V^{n-2} of semi-axes $\mu_1\ \mu_2\ \ldots\ \mu_{n-1}$. If σ varies, the locus of the $n-1$ pairs of points, situated in the normal to σ through the centre O of V^{n-1} at distances $\mu_1\ \mu_2\ \ldots\ \mu_{n-1}$ from O, is the wave-space W_2^{n-1} represented by

$$\sum_{i=1}^{i=n} x_i^2/(k^4 - a_i^2 \Sigma x_i^2) = 0.$$

The spaces W_1^{n-1} and W_2^{n-1} coincide in the same quadratic space W^{n-1} of degree and class $2(n-1)$.

3. In various publications Professor Mannheim has introduced a third mode of generation of the wave-surface. The object of this paper is to give an analytical demonstration of this third mode of generation, and to solve the question whether it is capable of as simple an extension to n-dimensional space.

According to the mode of generation in view, the wave-surface is the locus of the point P that admits with respect to a given ellipsoid L an enveloping cone, one of the principal sections of which is a right angle.

If we put $L \equiv x_1^2/b_1^2 + x_2^2/b_2^2 + x_3^2/b_3^2 - 1 = 0$, the enveloping cone with the vertex P $(y_1,\ y_2,\ y_3)$ is represented by

$$\Sigma\left(\frac{y_2^2}{b_2^2} + \frac{y_3^2}{b_3^2} - 1\right)\frac{x_1^2}{b_1^2} - 2\Sigma\frac{y_2 y_3}{b_2^2 b_3^2} x_2 x_3 = 0$$

referred to parallel axes through P.

This corresponds to the symbolical form $(c_1 x_1 + c_2 x_2 + c_3 x_3)^2 = 0$, if we put

$$c_{11} = \frac{1}{b_1^2}\left(\frac{y_2^2}{b_2^2} + \frac{y_3^2}{b_3^2} - 1\right), \qquad c_{23} = -y_2 y_3/b_2^2 b_3^2, \text{ etc.}$$

Now, according to the conditions of the problem the equation in S,—

$$D \equiv \begin{vmatrix} c_{11}-S & c_{12} & c_{13} \\ c_{21} & c_{22}-S & c_{23} \\ c_{31} & c_{32} & c_{33}-S \end{vmatrix} = 0$$

must have two roots whose sum is zero.

If D_0 is the value of D for $S=0$, and C_{11}, C_{22}, C_{33} be the minors of D_0 with respect to c_{11} c_{22} c_{33}, the condition is

$$(c_{11}+c_{22}+c_{33})(C_{11}+C_{22}+C_{33}) = D_0.$$

In this way we find after some calculation

$$\Sigma(b_2^2+b_3^2)y_1^2 \ \Sigma y_1^2 - \ \Sigma(b_2^2+b_3^2)(2b_1^2+b_2^2+b_3^2)y_1^2$$
$$+\Sigma b_1^2 \ \Sigma b_2^2 b_3^2 - b_1^2 b_2^2 b_3^2 = 0,$$

which reduces to

$$\sum_1^3 y_i^2/(k^4 - a_i^2 \Sigma y_i^2) = 0$$

by the substitution

$$b_2^2+b_3^2 = k^4/a_1^2, \quad b_3^2+b_1^2 = k^4/a_2^2, \quad b_1^2+b_2^2 = k^4/a_3^2.$$

So we find that the new surface coincides with W_1 and W_2, if L has the equation

$$x_1^2\Big/\left(\frac{1}{a_2^2}+\frac{1}{a_3^2}-\frac{1}{a_1^2}\right) + x_2^2\Big/\left(\frac{1}{a_3^2}+\frac{1}{a_1^2}-\frac{1}{a_2^2}\right) + x_3^2\Big/\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}-\frac{1}{a_3^2}\right) - k^4 = 0.$$

4. Let us proceed now to the space S^n and seek the locus of the point P that admits with respect to a given quadratic space $L^{n-1} \equiv \sum_{i=1}^{i=n} \frac{x_i^2}{b_i^2} - 1 = 0$ an enveloping cone, one of the principal sections of which is a right angle.

If we take parallel axes passing through $P(y_1, y_2 \ldots y_n)$, the equation of the enveloping cone of vertex P is

$$\Sigma\left(\frac{y_2^2}{b_2^2}+\frac{y_3^2}{b_3^2}+ \ \ldots \ +\frac{y_n^2}{b_n^2}-1\right)\frac{x_1^2}{b_1^2} - 2\Sigma\frac{y_i y_j}{b_i^2 b_j^2}x_i x_j = 0.$$

This appears in the symbolical form

$$(c_1x_1 + c_2x_2 + \ . \ . \ . \ + c_nx_n)^2 = 0$$

by the substitution

$$c_{11} = \frac{1}{b_1^2}\left(\frac{y_2^2}{b_2^2} + \frac{y_3^2}{b_3^2} + \ . \ . \ + \frac{y_n^2}{b_n^2} - 1\right), \quad c_ic_j = -y_iy_j/b_i^2b_j^2, \text{ etc.}$$

Now, according to the condition of the problem, the equation

$$D \equiv \begin{vmatrix} c_{11} - S & c_{12} & \vdots & c_{1n} \\ c_{21} & c_{22} - S & \vdots & c_{2n} \\ \cdots & \cdots & \vdots & \cdots \\ c_{n1} & c_{n2} & \vdots & c_{nn} - S \end{vmatrix} = 0$$

must have two roots in S whose sum is zero.

For if S_1, S_2 ... S_n be the roots, the equation of the enveloping cone with reference to its axes of symmetry is $\Sigma S_ix_i^2 = 0$ and the principal plane section $S_ix_i^2 + S_jx_j^2 = 0$ situated in the plane X_iOX_j is a right angle $x_i^2 - x_j^2 = 0$ provided $S_i + S_j = 0$.

If we suppose the roots to be

$$S_0, \quad S_0, \quad T_1, \quad T_2, \ . \ . \ . \ T_{n-2},$$

and represent respectively by λ, s_k, t_k the square of S_0 and the sum of the partial products k at a time of all the n roots, and of the $n - 2$ roots T, we have

$$\begin{aligned} s_1 &= t_1 \\ s_2 &= t_2 - \lambda \\ s_3 &= t_3 - \lambda t_1 \\ s_4 &= t_4 - \lambda t_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ s_{n-2} &= t_{n-2} - \lambda t_{n-4} \\ s_{n-1} &= \qquad - \lambda t_{n-3} \\ s_n &= \qquad - \lambda t_{n-2}. \end{aligned}$$

The elimination of t_1 t_2 . . . t_{n-2} gives the two equations

$$\left.\begin{aligned} s_{n-1} + \lambda s_{n-3} + \lambda^2 s_{n-5} + \ . \ . \ . \ . \ . \ &= 0 \\ s_n \quad + \lambda s_{n-2} + \lambda^2 s_{n-4} + \ . \ . \ . \ . \ . \ &= 0 \end{aligned}\right\} \quad . \quad . \qquad (1)$$

which can be treated by the dialytic method of Sylvester.

We may obtain the equations (1) in an easier manner. For, if the equation in S is written in the form

$$S^n \quad - s_1 S^{n-1} + s_2 S^{n-2} - s_3 S^{n-3} + \ldots\ldots = 0,$$

the conditions for the two solutions $+S_0$, and $-S_0$ are

$$S_0^n \quad \pm s_1 S_0^{n-1} + s_2 S_0^{n-2} \pm \ldots\ldots = 0$$

or

$$\left.\begin{array}{l} S_0^n \quad + s_2 S_0^{n-2} + s_4 S_0^{n-4} + \ldots\ldots = 0 \\ s_1 S_0^{n-1} + s_3 S_0^{n-3} + \quad \ldots\ldots\ldots\ldots = 0 \end{array}\right\},$$

a system which is identical with (1) as is proved by the substitution $S_0^2 = \lambda$.

The general result being too complicated for practical use, we consider the particular case $n = 4$. Then we have

$$s_3 + \lambda s_1 = 0 \qquad s_4 + \lambda s_2 + \lambda^2 = 0$$

and find

$$s_3^2 - s_1 s_2 s_3 + s_1^2 s_4 = 0.$$

Now the determinant

$$D_0 \equiv \begin{vmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{vmatrix}$$

leads to the notation

$$s_1 = c_{11} + c_{22} + c_{33} + c_{44}, \quad s_2 = \Sigma \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}, \quad s_3 = \Sigma \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = \Sigma C_{11},$$

$$s_4 = D_0,$$

where the quantities and minors following the Σ's belong to and have principal minors belonging to the principal diagonal of D_0.

Let P denote $y_1^2/b_1^2 + y_2^2/b_2^2 + y_3^2/b_3^2 + y_4^2/b_4^2 - 1$, then the form of D_0,

$$\begin{vmatrix} \frac{1}{b_1^2}\left(P - \frac{y_1^2}{b_1^2}\right), & -\frac{y_1 y_2}{b_1^2 b_2^2}, & -\frac{y_1 y_3}{b_1^2 b_3^2}, & -\frac{y_1 y_4}{b_1^2 b_4^2} \\ -\frac{y_2 y_1}{b_2^2 b_1^2}, & \frac{1}{b_2^2}\left(P - \frac{y_2^2}{b_2^2}\right), & -\frac{y_2 y_3}{b_2^2 b_3^2}, & -\frac{y_2 y_4}{b_2^2 b_4^2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

leads to the following values of s:

$$Bs_1 = \Sigma b_1^2 b_2^2 y_3^2 - \Sigma b_1^2 b_2^2 b_3^2, \qquad Bs_2 = P(\Sigma b_1^2 y_2^2 - \Sigma b_1^2 b_2^2)$$
$$Bs_3 = \quad P^2(\Sigma y_1^2 - \Sigma b_1^2) \quad , \qquad Bs_4 = -P^3, \text{ where } B = b_1^2 b_2^2 b_3^2 b_4^2.$$

Substitution of these values in the relation

$$s_3^2 - s_1 s_2 s_3 + s_1^2 s_4 = 0$$

gives

$$\begin{aligned} W_3^3 \equiv (\Sigma b_1^2 b_2^2 b_3^2 y_4^2 - b_1^2 b_2^2 b_3^2 b_4^2)(\Sigma y_1^2 - \Sigma b_1^2)^2 \\ - (\Sigma b_1^2 b_2^2 y_3^2 - \Sigma b_1^2 b_2^2 b_3^2)(\Sigma b_1^2 y_2^2 - \Sigma b_1^2 b_2^2)(\Sigma y_1^2 - \Sigma b_1^2) \\ - (\Sigma b_1^2 b_2^2 y_3^2 - \Sigma b_1^2 b_2^2 b_3^2)^2 = 0. \end{aligned}$$

This equation represents a tridimensional space W_3^3 of the sixth order, which contains only once the imaginary sphere at infinity common to all hyperspheres in S^4. Therefore it can not coincide with $W_1^3 = W_2^3$ of which that sphere is a double surface.

Of the space W_3^3 the bidimensional surface $s_1 = 0$, $s_3 = 0$ is a double surface, in other words W_3^3 passes two times through the intersection of the quadratic space $s_1 = 0$ and the hypersphere $s_3 = 0$; this double surface is not situated in a linear tridimensional space, as that of $W_1^3 = W_2^3$.

So we have shown that the third method of generation of the wave-surface, found by Professor Mannheim, admits a generalisation to n-dimensional space, but that this extension is of a more complicated character than that of the two ordinary modes of generation.

Sixth Meeting, May 13th, 1898.

J. B. Clark, Esq., M.A., F.R.S.E., President, in the Chair.

On the Second Solutions of Lamé's Equation

$$\frac{d^2U}{du^2} = U\left\{n(n+1)pu + B\right\}.$$

By Lawrence Crawford, M.A., B.Sc.

1. I consider here the second solutions corresponding to the solutions of the above equation (when n is an integer) in finite terms for special values of B. If U_n be such a solution, and F_n the corresponding second solution, we know that

$$F_n = (2n+1)U_n\int_0^u \frac{du}{(U_n)^2}.$$

2. U_n may be of one of four types, n being even or odd. Consider first the case of n even, $2m$ say, then the first type is

$$U_n = (pu - a_1)(pu - a_2) \;.\;.\;.\;.\; (pu - a_m),$$

where all the a's are different and no one coincides with an e, as I have proved in a former paper. *

The F_n corresponding to this is then

$$(2n+1)U_n\int_0^u \frac{du}{(pu-a_1)^2 \;.\;.\;.\;.\; (pu-a_m)^2};$$

proceed to the consideration of this integral.

Let $$\frac{1}{(pu-a_1)^2(pu-a_2)^2 \,.\,.\, (pu-a_m)^2} = \sum_{r=1}^{r=m}\left(\frac{A_r}{pu-a_r} + \frac{A'_r}{(pu-a_r)^2}\right),$$

then $$A'_r = \frac{1}{(a_r-a_1)^2(a_r-a_2)^2 \,.\,.\,.\, (a_r-a_m)^2}$$

and

$$A_r = \left[\frac{1}{p'u}\frac{d}{du}\left\{\frac{1}{(pu-a_1)^2 \,.\,.\, (pu-a_{r-1})^2(pu-a_{r+1})^2 \,.\,.\, (pu-a_m)^2}\right\}\right]_{pu=a_r}$$

$$= -\frac{2}{(a_r-a_1)^2 \,.\,.\, (a_r-a_m)^2}\left\{\frac{1}{a_r-a_1} + \frac{1}{a_r-a_2} + \,.\,.\, + \frac{1}{a_r-a_m}\right\}.$$

* "On the Factors of the Solutions in Finite Terms of Lamé's Equation," *Quarterly Journal of Pure and Applied Mathematics*, No. 114, 1897.

By differentiation of $\dfrac{p'u}{pu-a}$, it is easy to prove that

$$\int\frac{du}{(pu-a)^2} = -\frac{1}{4a^3-g_2a-g_3}\cdot\frac{p'u}{pu-a}$$
$$+\int\frac{2(pu-a)du}{4a^3-g_2a-g_3}-\frac{6a^2-\frac{1}{2}g_2}{4a^3-g_2a-g_3}\int\frac{du}{pu-a},$$

$\therefore$ we find

$$\int_0^u\frac{du}{U_n^2}=\sum_{r=1}^{r=m}\left[-\frac{A'_r}{4a_r^3-g_2a_r-g_3}\cdot\frac{p'u}{pu-a_r}-\frac{2A'_r(\zeta u+a_ru)}{4a_r^3-g_2a_r-g_3}\right.$$
$$\left.+\left\{A_r-\frac{A'_r(6a_r^2-\frac{1}{2}g_2)}{4a_r^3-g_2a_r-g_3}\right\}\int_0^u\frac{du}{pu-a_r}\right].$$

I proceed now to prove that $A_r-\dfrac{A'_r(6a_r^2-\frac{1}{2}g_2)}{4a_r^3-g_2a_r-g_3}=0,$

noting that $\qquad A_r=\left[\dfrac{1}{p'u}\cdot\dfrac{d}{du}\left(\dfrac{\overline{pu-a_r}}{U_n^2}\right)\right]_{pu=a_r}.$

In the differential equation for U_n, put $U_n=R(pu-a_r)$, then
$$\frac{d^2U_n}{du^2}=(pu-a_r)\frac{d^2R}{du^2}+2\frac{dR}{du}p'u+Rp''u=\big(n(n+1)pu+B\big)(pu-a_r)R$$
$\therefore$ when $pu=a_r$, as R and therefore $\dfrac{d^2R}{du^2}$ is not then infinite,
$$\left[2\frac{dR}{du}p'u+Rp''u\right]_{pu=a_r}=0.$$
But
$$A_r=\left[\frac{1}{p'u}\,\frac{d}{du}\left(\frac{1}{R^2}\right)\right]_{pu=a_r}=\left[-\frac{2\dfrac{dR}{du}}{R^3p'u}\right]_{pu=a_r},\qquad A'_r=\left[\frac{1}{R^2}\right]_{pu=a_r},$$
$\left[2\dfrac{dR}{du}p'u+Rp''u\right]_{pu=a_r}=0,$ and $\left[R^3p'^2u\right]_{pu=a_r}$ is not equal to 0,
$$\therefore\quad\left[\frac{2\dfrac{dR}{du}}{R^3p'u}+\frac{p''u}{R^2p'^2u}\right]_{pu=a_r}=0,$$
i.e. $\quad\left[A'_r\dfrac{p''u}{p'^2u}-A_r\right]_{pu=a_r}=0,$

i.e. $\quad A_r-A'_r\cdot\dfrac{6a_r^2-\frac{1}{2}g_2}{4a_r^3-g_2a_r-g_3}=0$

$\therefore$ all such terms as $\displaystyle\int_0^u \frac{du}{pu - a_r}$ do not appear in $F_n(u)$,

and
$$\int_0^u \frac{du}{U_n^2} = -\sum_{r=1}^{r=m} \frac{A'_r}{4a_r^3 - g_2a_r - g_3}\left(2\zeta u + 2a_r u + \frac{p'u}{pu - a_r}\right)$$
$$= Cu + D\zeta u - p'u \,.\, \Sigma \frac{A'_r}{(4a_r^3 - g_2a_r - g_3)(pu - a_r)}$$
$$= Cu + D\zeta u + \frac{p'u f(pu)}{U_n}$$

where C, D are constants, $f(pu)$ is an algebraic integral function of pu, the highest power involved being $p^{m-1}u$, and D is twice the coefficient of $p^{m-1}u$ in $f(pu)$,
$$\therefore \quad F_n = (2n+1)\{p'uf(pu) + U_n(Cu + D\zeta u)\}.$$

3. I shall work out now the second solution when U_n is of one of the types for n odd, having an irrational factor $\sqrt{pu - e}$. Then if $n = 2m+1$, $U_n = \sqrt{pu - e}(pu - a_1)(pu - a_2) \ldots (pu - a_m)$, where all the a's are real and different and no one coincides with an e, as I have proved in the former paper already referred to.

Then
$$F_n = (2n+1)U_n \int_0^u \frac{du}{(pu - e)(pu - a_1)^2(pu - a_2)^2 \ldots (pu - a_m)^2},$$

and
$$\frac{1}{(pu - e)(pu - a_1)^2 \ldots (pu - a_m)^2} = \frac{C}{pu - e} + \sum_{r=0}^{r=m} \frac{A_r}{pu - a_r} + \sum_{r=0}^{r=m} \frac{A'_r}{(pu - a_r)^2},$$

where
$$C = \frac{1}{(e - a_1)^2(e - a_2)^2 \ldots (e - a_m)^2}, \quad A'_r = \frac{1}{(a_r - e)(a_r - a_1)^2 \ldots (a_r - a_m)^2},$$

and
$$A_r = \left[\frac{1}{p'u}\frac{d}{du}\left\{\overline{\frac{pu - a_r}{U_n^2}}^{\,2}\right\}\right]_{pu = a_r}.$$

By differentiating $\dfrac{p'u}{pu - e}$, it is found that
$$\int \frac{6e^2 - \frac{1}{2}g_2}{pu - e}du = 2\int (pu - e)du - \frac{p'u}{pu - e},$$

and with the result already quoted for $\int \frac{du}{(pu-a_r)^2}$, we have

$$\int_0^u \frac{du}{U_n^2} = -\frac{C}{6e^2 - \frac{1}{2}g_2}\left\{2\zeta u + 2eu + \frac{p'u}{pu-e}\right\}$$
$$+\sum_{r=1}^{r=m}\left[-\frac{A_r'}{4a_r^3 - g_2a_r - g_3}\cdot\frac{p'u}{pu-a_r} - \frac{2A'_r(\zeta u + a_r u)}{4a_r^3 - g_2a_r - g_3}\right.$$
$$\left.+\int_0^u \frac{du}{pu-a_r}\left\{A_r - \frac{A'_r(6a_r^2 - \frac{1}{2}g_2)}{4a_r^3 - g_2a_r - g_3}\right\}\right].$$

Just as in the previous case, it follows that $A_r - \frac{A'_r(6a_r^2 - \frac{1}{2}g_2)}{4a_r^3 - g_2a_r - g_3} = 0$ by the substitution in the differential equation of $R(pu-a_r)$ for U_n, hence

$$\int_0^u \frac{du}{U_n^2} = -\frac{C}{6e^2 - \frac{1}{2}g_2}\left\{2(\zeta u + eu) + \frac{p'u}{pu-e}\right\}$$
$$-\sum_{r=1}^{r=m}\frac{A_r'}{4a_r^3 - g_{2r} - g_3}\cdot\frac{p'u}{pu-a_r} - 2\Sigma\frac{A_r'}{4a_r^3 - g_2a_r - g_3}(\zeta u + a_r u)$$
$$= C'u + D\zeta u + \frac{p'uf(pu)}{(pu-e)(pu-a_1)\,..\,(pu-a_m)},$$

where C', D are constants, $f(pu)$ an algebraic integral function of pu, the highest power involved being $p^m u$,

$$\therefore \qquad F_n = (2n+1)\left\{\frac{p'uf(pu)}{\sqrt{pu-e}} + (C'u + D\zeta u)U_n\right\}.$$

4. Similar work may be done for all cases, and the general form is $F_n = (2n+1)\left\{\frac{p'uf(pu)}{g(pu)} + (Cu + D\zeta u)U_n\right\}$, where $f(pu)$ is an algebraic integral function of pu, the highest power involved being pu to the power, when n is even, $n/2$ or $(n-2)/2$, according as U_n has no irrational factor or one, and when n is odd, $(n-1)/2$ or $(n-3)/2$, according as U_n has an irrational factor $\sqrt{pu-e}$ or the factors $\sqrt{(pu-e_1)(pu-e_2)(pu-e_3)}$, $g(pu)$ is the irrational factor, if any, in U_n, and C, D are constants, functions of the roots of the equation $U_n = 0$, regarded as an equation in pu.

5. The forms for the second solutions are found in Halphen, *Fonctions Elliptiques*, Vol. II., pp. 483–5, but it is interesting to see that they can be worked out in this way by direct integration.

On the Insolation of a Sun of Sensible Magnitude.

By A. RITCHIE SCOTT, B.Sc.

[ABSTRACT.]

Many theories have been advanced to account for a presumed uniformity in the temperature of the earth in past ages, and one of the most recent is that advanced by Sir John Murray in the Summary Volumes of the *Challenger Report*. A careful study of the distribution of marine fauna showed the existence of remarkably similar organic forms in the Arctic and Antarctic regions which were entirely unknown in intermediate waters. The existence, also, of ancient corals in Polar seas points to a high Polar temperature at some remote period of the earth's history. These facts combined lead us to consider it probable that in some past age the earth's air temperature was high and uniform. Sir John Murray, founding upon a paper by M. Blandet, suggests that this may have been due to the fact that life may have been possible on the earth long before the sun, in its cosmic development, had shrunk to its present dimensions. At present, at the solstice the sun only shines up to the Polar Circle at one of the poles, but the paleocosmic sun may have been so large that at the solstice the limb of the sun might still have shone on the pole and thereby kept up the Polar temperature. The present paper is an attempt to work out, as far as possible, the redistribution of temperatures under such circumstances.

For simplicity, we will assume that the large sun is of such a size that at the solstice its upper limb just illumines the poles—*i.e.*, its angular semi-diameter will be 23° 27′. We may also assume that it emits the same total energy as at present.

While the whole sun is above the horizon the rate of insolation will obviously be independent of the size of the solar disc. When, however, the sun is setting, the upper limb of the sun will continue to insolate any given point on the earth's surface for a considerable time after the centre of the solar disc has disappeared, and the same phenomenon will occur in opposite order during sunrise. During such intervals the given point on the earth's surface will receive a quantity of energy beyond what it gets at present from a (practically) point sun. Putting this otherwise, the large sun will

shine round the hemisphere further than at present ($23\frac{1}{2}°$ on our present assumption). In order to calculate the rate of insolation from the sun at any given altitude, the solar disc was divided into elementary strips parallel to the horizon; the insolation from each strip is proportional to the length of the strip multiplied by the cosine of the angle of inclination. The insolation from the whole sun was then got by integrating from the horizon to the upper limb of the sun. The integral being irreducible, the integration was graphical.

A formula was next investigated, giving the relation between the height of the sun and the time of day. The hours after noon were then laid down as abscissæ and the rate of insolation at the calculated height as ordinates, and a curve drawn. The area of the curve by planimeter gave the energy incident from noon to sunset.

In order to get comparable results, the insolation due to a point sun was plotted simultaneously with that of the sun of sensible magnitude, and measured under the same conditions.

LAT.	SUMMER SOLSTICE.		EQUINOX.		WINTER SOLSTICE.	
	LARGE SUN.	POINT SUN.	LARGE SUN.	POINT SUN.	LARGE SUN.	POINT SUN.
0°	950	920	1027	1000	950	920
10°	1038	1013	1009	984	818	793
20°	1113	1087	965	940	685	661
23° 27′	1126	1098	934	908	629	602
30°	1160	1130	892	864	535	503
40°	1183	1146	798	766	385	350
45°	1205	1158	725	694	311	264
50°	1203	1160	686	649	248	190
60°	1203	1129	521	471	127	75
66° 33′	1213	1146	474	406	73	0
70°	1215	1169	421	344	55	0
80°	1242	1234	324	163	20	0
90°	1267	1267	283	0	0	0

This table sums up the results of the work. A complete discussion would involve the summing of the daily insolation at each latitude

throughout the whole year, an amount of work which did not seem warranted by the importance of the calculation, particularly as the broad result is sufficiently indicated by these numbers.

The step from incident energy to air temperature is difficult, if not impossible, in the present state of meteorology; nevertheless, the comparatively small increase in the amount incident radiant energy, on account of the size of the sun, does not seem to warrant our accepting a sun of sensible magnitude as a sole cause for uniformity of temperature.

Some other interesting figures may be readily got from our data. Since the sun's polar distance and zenith distance are equal at the pole, the zenith distance may be approximately expressed as a simple function of the time of year. Plotting the time (in days from equinox) as abscissæ and the insolation at each date as ordinates, we find graphically that the energy incident at the pole during one whole year under the large sun would be to that at present as 1173 : 1000.

Again, consider any point on the earth's surface with the sun in the zenith; here the rate of insolation is a maximum. Around this point we may draw concentric small circles at all points, of any one of which the rate of insolation is the same. Imagine now the energy incident throughout any short time to be piled up on the earth's surface; we will now have a solid of revolution whose volume will be proportional to the rate at which the earth is intercepting radiant energy. The ratio of the rates at which energy is intercepted from the large sun and the small sun respectively is 1072 : 1000.

The Singular Solutions of a Certain Differential Equation of the Second Order.

By Mr Hugh Mitchell.

The subject of the Singular Solutions of Differential Equations of higher orders than the first is not touched in the ordinary text-books. Their existence, for instance, is not mentioned by Forsyth in his Treatise. This is probably due to the fact that, while in the case of equations of the first order a theory has been developed by Cayley and others which connects the singular solution in a geometrical manner with the ordinary solutions (the singular solution being, of course, the envelope of the ordinary solutions), in the case of equations of, say, the second order no corresponding theory exists—at any rate, no corresponding theory has yet been developed. Our only guide in the subject at present is Cauchy's Existence Theorem, which points out where we are to look for singular solutions.

As few illustrations of the subject have been given, Professor Chrystal thought that the following one might be of interest and not unworthy of your notice.

The equation whose singular solutions are to be examined is one which is given by Forsyth in his treatise, chap IV. Miscell. Ex. 1 (IX.)

$$(y'^2 - yy'')^2 = n^2(y^2 + ay''^2) \quad - \quad - \quad - \quad (1)$$

This equation does not contain the independent variable x, so writing in the usual manner $y'\dfrac{dy'}{dy}$ for y''

we get
$$y'^2\left\{\left(y' - y\frac{dy'}{dy}\right)^2 - n^2\left(1 + a^2\left(\frac{dy'}{dy}\right)^2\right)\right\} = 0$$

or
$$y' - y\frac{dy'}{dy} = \pm n\left(1 + a^2\left(\frac{dy'}{dy}\right)^2\right)^{\frac{1}{2}} \quad - \quad - \quad - \quad (2)$$

This is Clairaut's form if we regard y' as dependent and y as independent variable. Hence a first integral is

$$y' - yA = \pm n(1 + a^2A^2)^{\frac{1}{2}}$$

Integrating the linear equation we have

$$y = \pm\frac{n}{A}(1 + a^2A^2)^{\frac{1}{2}} + Be^{Ax} \quad - \quad - \quad - \quad \text{(I.)}$$

This is the ordinary complete primitive of the equation containing two arbitrary constants.

But since (2) is Clairaut's form, it has also a singular first integral, namely, its discriminant with regard to $\frac{dy'}{dy}$

$$a^2y'^2 - a^2n^2 + y^2 = 0 \quad - \quad - \quad - \quad (3)$$

The integral of this is

$$y = na\sin\left(\frac{n}{a} + C\right) \quad - \quad - \quad - \quad \text{(II.)}$$

Before going further, let us see what these two solutions correspond to geometrically.

The first is simply a twofold family of graphs of the exponential function, being always of the form $y = S + e^{Tx}$. When A has a definite value and B varies, the curves form spreads issuing from the ends of the lines $y = \pm\frac{n}{A}(1 + a^2A^2)^{\frac{1}{2}}$.

The axis of the spread is itself a solution, and corresponds to $B = 0$. If we vary A we vary the position of the axis of the spread. When $A = \infty$ the axis is $y = na$, when $A = 0$ the axis is $y = \infty$. Thus the axis of the spread of curves never lies between $y = \pm na$.

The second solution (II.) is simply a one-fold family of curves of sines lying between the two lines $y = \pm na$.

Now the solution (I.) contains two arbitrary constants, and is therefore the complete primitive and the solution whose existence for non-critical points is guaranteed by Cauchy's Existence Theorem. The solution (II.) is not a particular case of the first, and in fact has no member in common with it. It is therefore a singular solution.

In contrast with the theory of singular solutions of equations of the first order, two things are to be noticed. Firstly, that there is not merely one singular solution, but a one-fold infinity of singular solutions; and, secondly, that the singular solutions cannot, of course, be regarded as envelopes of the ordinary solutions.

The existence of this family of singular solutions was suggested by the ordinary process of solving the equation. But we may look at the point of view of Cauchy's Existence Theorem, and examine the various critical loci.

Arranging the equation in powers of the highest differential coefficient y'' the equation is of the form

$$y''^2(y^2 - n^2a^2) - 2yy'^2y'' + y'^2(y'^2 - n^2)$$
$$Py''^2 + Qy'' + R = 0$$

and the critical loci are the discriminant with respect to y'' and the loci which make the coefficients P, Q and R vanish.

The y''-discriminant is

$$y'^2(y'^2a^2 + y^2 - n^2a^2) = 0.$$

The factor in brackets is (3) which gave the solution (II.)

$$y'^2 = 0 \quad \text{gives} \quad y = D \quad - \quad - \quad - \text{(III.)}$$

which on trial is found to be a solution of the equation, and is thus a second family of singular solutions. It is to be noticed, however, that outside of the two lines $y = \pm na$ its members are also members of the complete primitive. But part of the family is not included in the complete primitive, and since it consists of elements (x, y, y') for which Cauchy's normal form ceases to be synectic, all the members of the family may be considered singular solutions.

We next examine the locus $R = 0$

$$\text{that is} \quad y'^2 - n^2 = 0$$
$$\text{that is} \quad y = \pm nx + E \quad - \quad - \quad - \text{(IV.)}$$

This on trial is found to satisfy the equation, and is thus a third family of singular solutions.

There remains to be considered the locus $P = 0$

$$\text{or} \quad y^2 - n^2a^2 = 0.$$

This locus might have come in otherwise. It is, in fact, the envelope singular solution of (3), which was itself a singular first integral of the equation. It is therefore a solution of the equation. It is, however, a limiting form of the complete primitive as well.

This example shows in a remarkable manner how far from being the only solution of a differential equation the complete primitive

containing the full number of arbitrary constants may be. The critical loci, in fact, will be in general differential equations of an order lower by one than the differential equation itself, and they may furnish singular solutions containing that number of arbitrary constants.

In the case of equations of the first order, the critical locus has been analysed into cusp-locus, tac-locus, and envelope singular solution, and criteria for each have been given. No such analysis of the critical locus of an equation of, say, the second order has been attempted, and no criterion for the existence of singular solutions given. Still, it is possible that in a physical problem it might be well to bear in mind that, outside of the most general solution of an equation, solutions of a less degree of generality might also exist, and might be of a very different form.

Seventh Meeting, June 10th, 1898.

Dr MACKAY in the Chair.

Permutations: Alternative Proofs of Elementary Formulas.

By R. F. MUIRHEAD, M.A., B.Sc.

1. The number of the r-permutations of n things is the same as the number of ways of adding r things to a row which originally contains $n-r$ other things.

For suppose the n letters A, B, C . . . N to be placed in a row, then each permutation consisting of r letters may be indicated by placing below those r letters in the row, the digits 1, 2, 3 . . . r, to indicate the order they have in the permutation. We may suppose zeros placed below all the remaining $n-r$ letters. Thus it is clear that the number of the r-permutations of n things is the same as the number of ways in which r numbers can be added to a row originally containing $n-r$ zeros.

2. Hence we can prove the formula

$$_nP_r = n(n-1)(n-2)\ .\ .\ .\ (n-r+1).$$

For starting with $n-r$ zeros, we can put the first number at either end of the row, or in any of the $n-r-1$ places between two successive zeros, *i.e.*, we can add the first number to the row in $n-r+1$ different ways. The next number can then be added in $n-r+2$ ways, giving a total of $(n-r+1).(n-r+2)$ ways of adding two things to the row. Proceeding thus, we see that the total number of ways of adding r things to the row which originally contained $n-r$ is $(n-r+1)(n-r+2)\ .\ .\ .\ n$, which may also be written $n(n-1)(n-2)\ .\ .\ .\ (n-r+1)$.

This then is also the number of the r-permutations of n letters, all different.

3. The number of ways in which two rows of things containing p and q things respectively, can be combined into a single row containing them all, the order of the things in each component set being unaltered, is $= {}_{p+q}C_p$. For it is clearly the same as the number of ways of choosing p out of $p+q$ places for the things of the first set. Of course it is also $= {}_{p+q}C_q$.

Cor. The result is obviously the same as the number of ways of arranging p like things A, A . . . in a row along with q like things B, B . . . ; for the order of the A's amongst one another and that of the B's amongst one another, being indistinguishable, it is the same as if they had a fixed order.

4. The number of ways in which p A's, q B's, r C's and t other different letters can be arranged in order is $\dfrac{(t+p+q+r+\ldots)!}{p!q!r!\ldots}$.

For first the t different ways can be arranged in $t!$ orders. Taking any one of these, the p A's can be added in ${}_{p+t}C_p$ or $\dfrac{(p+t)!}{t!p!}$ ways to this row. Hence the total number of ways of forming a row consisting of the p A's and the t other things is $t!\dfrac{(p+t)!}{t!p!}$. Adding next the q B's and so on, we finally get at the result

$$t!.\frac{(t+p)!}{t!p!}.\frac{(t+p+q)!}{(t+p)!q!}.\frac{(t+p+q+r)!}{(t+p+q)!r!}\ldots$$

which simplifies at once to

$$\frac{(t+p+q+r+\ldots)!}{p!q!r!\ldots}, \qquad i.e. \quad \frac{n!}{p!q!r!\ldots}$$

where n is the total number of the things.

On the Transformation-Sequence.

By R. F. Muirhead, M.A., B.Sc.

This paper is a continuation of my paper, "On a Method of Studying Displacement," in last year's *Proceedings*. In that paper I showed how the chief theorems as to the displacements of rigid bodies could be simply demonstrated by the use of what I called a *Displacement-chain* or *Displacement-sequence*.

In this paper I complete the treatment of displacement accompanied by perversion, by adding an investigation of the three-dimensional perversion-sequence; and thereafter I show how the sequence method may be extended to other transformations besides those corresponding to rigid displacements and perversions.

From one point of view we may regard a displacement of a rigid body as a change by which every point in the body is transformed into another point of space, and by imagining the body unlimited in extent, we get the idea of a transformation by which we pass from any one point of space to some point which is in general different from it.

In the same way, starting with two bodies F and F_1, such that F_1 is the *perverted* reproduction of F, *i.e.*, such that F_1 is related to F as the right hand to the left, or as an object to its image in a plane mirror (but not necessarily in the special relative position which object and image occupy)—and supposing each of these bodies indefinitely extended—we get a space-transformation which is analogous to a rigid-displacement-transformation, but differs from it in the peculiar way indicated.

The nature of the perversion-sequence in two dimensions was discussed in the previous paper: let us now study it in three dimensions.

Let ABCDE (Fig. 15) be a perversion-sequence, with reference to a body F and its perversion F_1; so that A in F corresponds to B in F_1, B in F to C in F_1, and so on. Join P, Q, R, S the mid-points of AB, BC, CD, DE, and join AC, BD, CE, AE.

Then $AB = BC = CD = DE$; $AC = BD = CE$. By reasoning similar to that in the first half of p. 125, Vol. XV., we can show that the points E and A are on the same side of the plane BCD.

Hence the dihedral angles $A\overline{BD}C$ and $E\overline{DB}C$ (at the edge DB) are equal, since they are corresponding angles in the tetrahedrons ABDC and EDBC which are symmetrically equal to one another. Hence the planes ABD and EBD, making equal angles with the plane CBD on the same side of it, coincide; so that ABDE is a plane trapezium having AB = DE and BD || to AE.

QR which is || to BD, is || to AE

Also PQ is || to AC and RS to CE.

Thus PQ, QR, RS being parallel to three lines in the plane ACE, and conterminous, are coplanar.

Similarly Q, R, S, T are coplanar, ∴ PQRST are so. By induction we see that the mid-points of the successive steps AB, BC . . . in the perversion-sequence all lie in the same plane. This plane, then, is unaltered in position by the perversion, though its points are not unaltered.

Again, since A in F corresponds to B in F_1 the plane PQRS . . . will appear in F as seen from A, as the perverse of the corresponding plane in F_1 as seen from B. But A and B are on opposite sides of that plane. Hence the points of the plane PQRS . . . before and after transformation, are *congruent* when looked at from the same side of the plane.

Hence, as shown in Vol. XV., p. 120, there will be *one* point that will remain unaltered by the transformation in question.

Thus the most general perverted displacement in three dimensions may be analysed into a reflection with reference to a plane PQRS . . . , together with a rotation about a line perpendicular to that plane passing through the centre of the regular polygon PQRST . . .

The amount of the rotation is equal to the angle between PQ and QR, which is half that between AC and CE.

If we extended this method of analysis to perverted displacement in n dimensions we should find that in general a certain flat space of $n-1$ dimensions in the body of reference is unaltered in position but transformed so as to become congruent with itself, so that such a perverted displacement would be equivalent to a *reflection* in that $(n-1)$-flat together with a displacement of the ordinary sort by which the points of the $(n-1)$-flat would move in their most general manner consistent with the maintenance of that $(n-1)$-flat in its original position.

Let us now generalize by one degree, the space transformation associated with a displacement (with or without perversion), by supposing the body F to be not merely displaced, but also changed in magnitude, its shape however remaining unchanged. This amounts to the condition that the line joining each pair of points of F is altered in the same ratio, or that straight lines remain straight lines, and angles are unaltered in magnitude.

This we may call a Conformal Transformation. Let then ABCDE . . . (Fig. 16) be a Conformal Transformation-sequence, and first let us suppose it without perversion, and confined to two-dimensional space.

Since ABC transforms into BCD, the triangles are similar, and in fact it is obvious that ABCDE . . . forms an equiangular polygon, whose sides are in continued proportion. The eye sees at once that by producing the sequence indefinitely,* we should get indefinitely near to a point O which would be unaltered by the transformation, and which would be the pole of an equiangular spiral circumscribed about the polygon.

But the point O may be got more directly by describing on AB as chord, a circle which will touch BC at B, and lie on the same side of BC as AB does; and another circle on BC as chord, touching AB on the side on which C lies. These circles will cut again in a point O, which will correspond to itself, since the triangles OAB, OBC are obviously similar.

Thus in Conformal Transformation, there is one point O which is unaltered. This is what is called the Centre of Similitude of F and F′ in the theory of two bodies directly similar in one plane. If we join O to A, B, C, D . . . and cut off from OB, OC, OD . . . lengths O*a*, O*b*, O*c* . . . equal respectively to OA, OB, OC . . . we see that the transformation in question may be analysed into a rotation about O through an angle equal to AOB, (bringing ABC . . . to *abc* . . .) together with a radial contraction or expansion by which each radius from O is changed in the same ratio (bringing *a b c* . . . to BCD . . .).

Note that *three* points A, B, C of a sequence are sufficient to

* This is true if the ratio AB : BC is greater than unity; on the contrary supposition we should have to trace the sequence *backwards* to get towards O. This note will apply to later passages: but the application may be left to the reader.

determine the transformation completely. Suppose, however, two similar triangles LMN and L′M′N′ are given, and it is required to find their centre of similarity. Join LL′, ML′, and on L′M′ describe a △ L′M′L″ directly similar to the △ LML′. Obviously LL′L″ form a sequence, and the preceding construction for O can therefore be applied.

Next suppose F′ to be perversely similar to F. The corresponding space-transformation may be called a perversely-conformal transformation.

The transformation-sequence in this case (see Fig. 17) is clearly a zig-zag equiangular figure with sides in continued proportion, and parallel alternately, and the construction evidently tends towards a certain point O which would be unaltered by the transformation.

Let us divide AB, BC, CD ... in P, Q, R ... in the same ratio as AB : BC or BC : CD.

$$\text{Then}\quad AP : AB :: BQ : BC \quad \therefore\ AP : BQ :: AB : BC$$
$$:: AP : PB$$

$$\text{Hence}\quad PB = BQ \quad \therefore\ \angle BPQ = \angle BQP.$$

From this it follows easily that $\angle BQP = \angle CQR$ and that P, Q, R are collinear, and lie in a straight line which passes through O, the intersection of AC and BD.

PQR ... forms a sequence of collinear points whose distances from O are changed by the transformation in a fixed ratio, while the position of the line as such is unaltered.

It is obvious that the interior bisectors of such angles as ABC, BCD are all parallel, and unaltered in direction by the transformation.

If YOY′ be drawn through O perpendicular to PQR...O, we see that the transformation is equivalent to a reflection in the line OP, together with a linear contraction or expansion to or from OP, and another linear contraction or expansion to or from OY; or, more briefly, to a reflection in OP and a radial contraction or expansion to or from the centre O, by which each radius is changed in the same ratio.

Consider now the *direct* Conformal Transformation in *three* dimensions, by which a body F is transformed into a body F′ similar to F and similarly situated on its base.

The Transformation-sequence ABCDE ... in this case is a skew

equiangular polygon whose sides are in continued proportion, and whose successive planes ABC, BCD, CDE, etc., make equal angles in pairs, but so that A and E are on opposite sides of the plane BCD.

In fact it is a kind of skew rectilinear spiral, and obviously leads towards a *pole* which will be unaltered by the transformation.

By applying to it the construction and reasoning of p. 122, Vol. XV., it appears that as in the case of the Displacement-sequence the *directions* of the shortest distances between the interior bisectors of successive angles of the sequence, are unaltered, but they are no longer conterminous, nor in the same straight line. We see, however, that they are all in the same direction. All lines of F in this direction, and all planes perpendicular to this direction in F are unaltered in *direction* by the transformation.

It follows that if we take a sequence of such planes, the distance between any two in sequence will diminish in geometric progression as we go forwards (or backwards). Hence we shall approach a plane which is unaltered in position as well as in direction. When three of the planes in sequence are given we can get this unique plane by an obvious construction.

The points of this unique plane are in general changed by the transformation, but as it does not move out of itself, it can be made self-coincident in all its points by the direct conformal transformation in two dimensions.

We now see that the three-dimensional direct conformal transformation may be analysed into an expansion or contraction in a certain direction, leaving a certain plane fixed, together with a two-dimensional conformal transformation. Or we may state it as a rotation about a certain line which meets the unique plane in a fixed point, together with a radial expansion or contraction about that point, such that all radii are changed in the same ratio.

Thus one point, one line, and one plane retain their positions after transformation.

Take next, the Perverse Conformal Transformation. Let ABCDE... be a sequence, and P, Q, R, S... the points which divide AB, BC,... in the same ratio as AB to BC, etc.

As in the corresponding construction in two dimensions, it follows that $PB = BQ$, $QC = CR$, etc., and that $\angle BQP = \angle CQR$.

Hence the plane PQR makes equal angles with the planes

BPQ and CQR, and therefore the dihedral angles $P\overline{QR}B$ and $R\overline{PQ}B$ are supplementary.

But the tetrahedron CQRS is perversely similar to the tetrahedron BPQR. Hence the dihedral angles $C\overline{QR}S$ and $B\overline{PQ}R$ are equal. Therefore the dihedrals $C\overline{QR}S$ and $P\overline{QR}S$ are supplementary. Hence the planes PQR and QRS coincide.

Thus the points PQRS... form a coplanar sequence, and as in the corresponding case of perverted displacement, the plane containing them is unaltered by the transformation, and has its points *directly* conformal to one another.

We see, in fact, that the general Perversely-Conformal Transformation may be analysed into a reflection in the plane PQR... together with a rotation about an axis perpendicular to this plane, and through the pole of the rectilinear spiral PQR..., and an axial contraction towards or expansion from that pole as fixed centre.

Thus there is a point which remains unaltered, a line passing through it which retains its position, and a plane through the point and perpendicular to the line which also retains its position.

Next above the Conformal Transformation in generality, and including it as particular case, comes the Linear Transformation which distinguishes itself by the property that every straight line of F remains a straight line in F', while corresponding points are the intersections of corresponding lines.

In such a corresponding transformation parallel or concurrent lines would remain parallel or concurrent, hence an harmonic ratios of pencils and ranges would remain unaltered, and therefore the ratios between segments of the same line would remain unaltered.

For the general Linear Transformation in two dimensions, four points of a sequence would be necessary and sufficient for its determination, and the production of the sequence A, B, C, D might be thus carried out:

Join AC, BD and let them cut in L.
Divide BD in M so that BM : MD : : AL : LC.
Join CM and produce to E so that CM : ME : : BL : LD.
Then E is the next point of the sequence.

A similar construction would hold in three dimensions, five points of the sequence being given, the modification required for a perverse transformation being obvious in either case.

Elementary Notes.

By J. W. BUTTERS, M.A., B.Sc.

1. ON THE FACTORISATION OF A FUNCTION OF n VARIABLES.

The following examples illustrate a somewhat obvious extension of the method of factoring given in a former paper (*Proceedings*, Vol. XII., p. 32, q.v.). By means of it, if we are given any function of n variables, no one of which is of higher degree than the second, we can either find the factors of it, *or* prove that it has no factors with rational coefficients.

Example (1)

$$8x^2 + 115xy - 79xz + 52x + 42y^2 - 125yz - 35y + 63z^2 - 13z - 28.$$

This we may write as a trinomial in x, viz.,

$$8x^2 + (115y - 79z + 52)x + (42y^2 - 125yz - 35y + 63z^2 - 13z - 28).$$

Applying the method described in the former paper we have now to find two factors of the product of the coefficient of x^2 and the third term, such that their sum is the coefficient of x.

This requires us to find the factors of the third term, which we may write as a trinomial in y (say) viz.,

$$42y^2 - (125z + 35)y + (63z^2 - 13z - 28).$$

Applying the same method to this function of y, it is now necessary to find the factors of the third term, which may be written as a function of z. It is obvious that the process is quite general however many variables there may be. The work may be arranged as follows. The given function $=$

$$\begin{aligned} &8x^2 + (115y - 79z + 52)x + \{42y^2 - (125z + 35)y + (63z^2 - 13z - 28)\} \\ &= 8x^2 + (115y - 79z + 52)x + \{42y^2 - (125z + 35)y + (7z + 4)(9z - 7)\} \\ &= 8x^2 + (115y - 79z + 52)x + (3y - 7z - 4)(14y - 9z + 7) \\ &= (8x + 3y - 7z - 4)(x + 14y - 9z + 7). \end{aligned}$$

This example, however, is capable of a shorter treatment, for it is obvious that its factors are of the form

$$(ax + by + cz + d)(a'x + b'y + c'z + d').$$

Consider first, therefore, the factors of

$$8x^2 + 115xy + 42y^2$$

and we get $(8x + 3y)(x + 14y)$.

These give the first terms of the required factors.

Next consider

$$42y^2 - 125yz + 63z^2 \equiv \{\pm(3y - 7z)\}\{\pm(14y - 9z)\}$$

The upper signs are obviously the suitable ones and we now have

$$(8x + 3y - 7z \ . \ . \ . \ )(x + 14y - 9z \ . \ . \ . \ )$$

Next $\quad 63z^2 - 13z - 28 \equiv \{\pm(9z - 7)\}\{\pm(7z + 4)\}$

Whence, taking lower signs and second factor first, we get

$$(8x + 3y - 7z - 4)(x + 14y - 9z + 7).$$

It remains now to test whether these factors suit the terms which have not been used in their determination, viz., $-79xz + 52x - 35y$.

The following example shows that the method is not limited to functions, the form of whose factors may be determined by inspection.

Example (2)

$$\begin{aligned}
&3a^2b^2c - 9a^2bc^2 + a^2b - 3a^2c - 3abc^2 + 2ab^2c + ac + 2bc^2 \equiv \\
&(3a^2c + 2ac)b^2 - (9a^2c^2 - a^2 + 3ac^2 - 2c^2)b - (3a^2c - ac) \equiv \\
&ac(3a + 2)b^2 - (9a^2c^2 - a^2 + 3ac^2 - 2c^2)b - ac(3a - 1) \equiv \quad \cdot \ \text{(A)} \\
&\qquad \{a \, . \, b - (3a - 1)c\}\{c(3a + 2) \, . \, b + a\} \equiv \\
&\qquad (\ ab \ - 3ac - c)(3abc + 2bc + a).
\end{aligned}$$

Note: This has been arranged as a function of b; it might also have been arranged as a function of a or of c. At (A) we have to find two factors of $\quad ac(3a + 2) \, . - ac(3a - 1)$

whose sum is $\quad -(9a^2c^2 - a^2 + 3ac^2 - 2c^2)$.

By considering the coefficient of a^2 in this, we see that the factors are $\quad a^2 \;$ and $\; -c^2(3a + 2)(3a - 1)$.

So long as no variable is of higher degree than the second, the above methods are applicable, and they may be applied even when *some* of the variables are of higher degree than the second, provided we can arrange the function as a trinomial in one of the variables, and the last term and the coefficient of first term be each capable of being factored by known methods.

Example (3)

$$a^3c^2 + 2a^3c + a^3 + 4ab^2c + b^3 - b^3c^2$$

may be written $(a^3 - b^3)c^2 + (2a^3 + 4ab^2)c + (a^3 + b^3) =$

$$(a - b)(a^2 + ab + b^2) \cdot c^2 + (2a^3 + 4ab^2) \cdot c + (a + b)(a^2 - ab + b^2) =$$

$$\{(a^2 + ab + b^2) \cdot c + (a^2 - ab + b^2)\}\{(a - b) \cdot c + (a + b)\} =$$

$$(a^2c + a^2 + abc - ab + b^2c + b^2)(ac + a - bc + b).$$

2. On the use of the term "Produced."

One of the features distinguishiug Modern Geometry from Euclidean Geometry is that, by means of suitable conventions, its statements are made perfectly general, *e.g.*, *two straight lines meet in a point.* To one acquainted only with geometry as given in most editions of Euclid, there are two difficulties in this statement: first, parallel straight lines do not meet "even when produced ever so far both ways"; and, secondly, other straight lines may not meet unless *produced.* It would quite change the character of elementary geometry to adopt the convention whereby parallel lines are included in the above proposition, but the convention that straight lines are of unlimited length and do not need to be produced might with advantage be adopted, in teaching the "elements." It is obvious that by such a convention we may both make statements more comprehensive so as to include cases not formerly considered; and also group together cases *apparently* distinct, as has, indeed, been done in propositions in Book II. in many recent editions of Euclid.

But the convention has further advantages, as the following, out of a large number of examples occurring in recent examination papers, may show. They may easily be classified as those in which the term "produced" is (*a*) useless, or (*b*) misleading, or (*c*) wrong.

(*a*) "ABC is an isosceles triangle in which AB = AC. Through C, CD is drawn perpendicular to BC, meeting BA *produced* in D. Shew that A is the mid-point of BD."

Here it is useless to be given that CD meets BA produced, as this is capable of proof. There is, in fact, a redundancy in the data.

(*b*) "ABCK is a quadrilateral with AB = AC and angle K a right angle. E is the middle point of BC. From E perpendiculars are drawn to AK and KC *produced* meeting them in H and M respectively. Prove that AE : EC = AH : CM."

Here, any one beginning by drawing ABCK of Fig. 19 will have no difficulty, but what of the poor examinee who begins with ABCK of Fig 20? From E he can draw a perpendicular neither to AK nor to KC *produced.* The proposition, however, is equally true for both figures. By adopting the suggested convention we get rid both of the difficulty of drawing the figure and of the implication that the truth of the proposition is limited to the case where the perpendiculars lie on opposite sides of BC.

(*c*) "AB, BC are equal arcs of a circle and P is a point on the arc BC, show that BP bisects the angle contained by AP and CP *produced.*"

This proposition is not true unless AB and BC be minor arcs, as may be seen from Figs. 21 and 22, where AB is the arc ACB and BC is the arc BAC. The statement, however, that BP bisects one of the angles between AP and CP is always true, and, moreover, the limitation that P is a point on the arc BC may be removed, for evidently P may be any point on the circumference.

In connection with examples such as this it might be useful to adopt the convention of naming arcs and angles connected with a circle in the clockwise direction; so that AB and BA would denote conjugate arcs, and if O be the centre, AOB would stand on AB while BOA would stand on BA. It would then be unnecessary to distinguish reflex angles, and generality of proof would be gained.

[*Omitted by an oversight from the report of the December Meeting.*]

A General Method of Solving the Equations of Elasticity.

By John Dougall, M.A.

1. In the theory of Electrostatics, or of the Newtonian potential, there exists between two systems of potentiating matter, a well-known reciprocal relation, analytically expressed in the proposition known as Green's Theorm. By applying his theorem to the case when one of the systems is of the simplest possible character, namely, a mass concentrated at a single point, Green deduced a general method of solving the equation for the potential. The idea of a similar general method of dealing with the equations of Elasticity is due to Professor Betti, of Pisa, who has proved a reciprocal relation between two states of strain of an elastic solid, analogous to the relation in Electrostatics referred to. Following the example of Green, Betti considers what his theorem becomes when one of the states of strain belongs to one or other of certain very simple types, and obtains results which may be applied to the solution of the elastic equations. Strangely enough, however, Betti does not include among his simple types the type which we should naturally take as fundamental, namely, the strain in an infinite solid due to a force applied at a single point.

The discussion of this type of strain, from the point of view of Betti's theorem, is the object of the present paper. General theorems are reached, in which Betti's results will be seen to be included as special cases, by a method which makes physical interpretation easy.

2. In order to arrive at Betti's theorem, we start from the known principle that the potential energy of an elastic solid strained at constant temperature is a function of the strain only, *i.e.*, is independent of the succession of steps by which the strain may be produced.

Let (u, v, w), (X, Y, Z), (F, G, H) denote respectively the displacements, components of bodily force per unit volume, and components of surface traction, in any state of equilibrium of an

elastic solid, referred to for brevity as the state u, v, w. Let the same letters with accents denote the corresponding quantities in a second state of equilibrium u', v', w'.

The solid may be brought from the state u, v, w to the state u', v', w', by continuous passage through the states $u+n(u'-u)$, $v+n(v'-v)$, $w+n(w'-w)$, where n varies from 0 to 1. The forces required to maintain the intermediate state corresponding to a particular value of n will be $X+n(X'-X)$, etc. The work done by the applied forces in changing the n state into the $n+dn$ state will be

$$\text{Volume integral of } \{X+n(X'-X)\}(u'-u)dn+\ldots+\ldots$$
$$+\text{Surface integral of } \{F+n(F'-F)\}(u'-u)dn+\ldots+\ldots.$$

Integrating with regard to n from 0 to 1, we obtain for the total work of the applied forces in changing the state u, v, w into the state u', v', w',

$$\text{Volume integral of } \tfrac{1}{2}(X'+X)(u'-u)+\ldots+\ldots$$
$$+\text{Surface integral of } \tfrac{1}{2}(F'+F)(u'-u)+\ldots+\ldots.$$

This must be equal to the excess of the potential energy in the state u', v', w', over that in the state u, v, w. We see that it is half the work done by the resultant of the initial and final force, acting through the increase of displacement.

In particular, if we take the state of zero potential energy to be the state of no strain, we see that the potential energy in any state is half the work done by the forces maintaining that state, acting over the displacements of that state.

Denoting the energy of any state by $W(u, v, w)$ let us take for initial state u, v, w and for final state $u'-u$, $v'-v$, $w'-w$.

Hence $\quad W(u'-u,\ v'-v,\ w'-w)-W(u,\ v,\ w)$

$$= \text{Volume integral of } \tfrac{1}{2}X'(u'-2u)+\tfrac{1}{2}Y'(v'-2v)+\tfrac{1}{2}Z'(w'-2w)$$
$$+\text{Surface integral of } \tfrac{1}{2}F'(u'-2u)+\tfrac{1}{2}G'(v'-2v)+\tfrac{1}{2}H'(w'-2w)$$

$$= W(u', v', w)$$
$$-\{\text{vol. int of } (X'u+Y'v+Z'w)+\text{surf. int. of } (F'u+G'v+H'w)\}.$$

Or

$$\text{Vol. int. of } (X'u+Y'v+Z'w)+\text{surf. int. of } (F'u+G'v+H'w)$$
$$=W(u,\ v,\ w)+W(u',\ v',\ w')-W(u'-u,\ v'-v,\ w'-w).$$

But clearly $W(u' - u, v' - v, w' - w) = W(u - u', v - v', w - w')$.
Hence, from symmetry

$$\text{Vol. int. of } (X'u + Y'v + Z'w) + \text{surf. int. of } (F'u + G'v + H'w)$$
$$= \text{Vol. int. of } (Xu' + Yv' + Zw') + \text{surf. int. of } (Fu' + Gv' + Hw').$$

This last equality is the reciprocal theorem discovered by Prof. Betti, which will be referred to simply as Betti's Theorem.

It may be stated :—If two states of equilibrium of an elastic solid be taken, then the work done by the forces of the first state acting over the displacements of the second, is equal to the work done by the forces of the second state, acting over the displacements of the first.

3. The differential equations satisfied by the components of stress at any point of an elastic solid are obtained as conditions of equilibrium of any portion of the solid. In cases where the stresses are discontinuous at a surface S within the solid, these equations have to be supplemented by a surface condition.

Discontinuity in the stresses at S may conceivably arise from

(*a*) Application at the surface S of external force having a finite resultant per unit area.

(*b*) Abrupt change at S in the values of the elastic constants, and,

(*c*) So far at least as appears at first sight, abrupt change at S in the applied bodily force per unit volume.

Consider the equilibrium of a portion of the solid including within it part of the surface S, and bounded by a closed surface S′.

The surface S separates a region O from a region I of the solid; let dO, dI denote an element of volume in these respective regions; let l, m, n denote the direction cosines of the normal at any point of S, drawn from I to O; and l', m', n' the direction cosines of the outward normal at any point of S′.

Let A, B, C be the components of the force per unit area applied at S; X, Y, Z the components of the force per unit volume applied at any point of the solid.

Resolving the forces parallel to the axis of x, and using the notation of Lord Kelvin and Tait for the components of stress, we have

$$\iint (l'P + m'U + n'T)dS' + \iint A dS + \iiint X dO + \iiint X dI = 0$$

i.e. $$\iiint\left(\frac{dP}{dx} + \frac{dU}{dy} + \frac{dT}{dz} + X\right)dO + \iiint\left(\frac{dP}{dx} + \frac{dU}{dy} + \frac{dT}{dz} + X\right)dI$$

$$+ \iint \{(lP + mU + nT)_O - (lP + mU + nT)_I + A\}dS = 0.$$

The volume integrals vanish separately from the conditions of equilibrium of a portion of the solid wholly within either region; we have then the surface condition

$$(lP + mU + nT)_O - (lP + mU + nT)_I + A = 0,$$

and similarly two other conditions of like form, which must hold at every point of S.

By $(lP + mU + nT)_O$ is meant the value of $lP + mU + nT$ at S, measured in the region O, and similarly with $(lP + mU + nT)_I$.

The vanishing of the couples gives no new condition.

4. Confining ourselves now to the case of a solid, homogeneous and isotropic throughout, and subjected to no *superficial* internal applied force; that is, considering only, among possible sources of discontinuity of stress, discontinuity in the applied force per unit volume, we have the conditions at the surface S

$$(lP + mU + nT)_O - (lP + mU + nT)_I = 0$$
$$(lU + mQ + nS)_O - (lU + mQ + nS)_I = 0$$
$$(lT + mS + nR)_O - (lT + mS + nR)_I = 0.$$

These conditions are clearly satisfied if the first derivatives of u, v, w are continuous at S. This sufficient condition can be shown to be also necessary. For, in the first place, u, v, w are themselves continuous, since the solid is not to be ruptured. Hence the rate of variation of u, v, w per unit length in any direction lying in the surface must be continuous.

Denoting, then, by $\left(\frac{d}{d\nu}\right)_{\mathrm{O}}$, $\left(\frac{d}{d\nu}\right)_{\mathrm{I}}$, rate of variation per unit length in the direction l, m, n of the normal in the respective regions O, I, we have

$$\left(\frac{du}{dx}\right)_{\mathrm{O}} - \left(\frac{du}{dx}\right)_{\mathrm{I}} = \left(l\frac{du}{d\nu}\right)_{\mathrm{O}} - \left(l\frac{du}{d\nu}\right)_{\mathrm{I}}$$

$$\left(\frac{du}{dy}\right)_{\mathrm{O}} - \left(\frac{du}{dy}\right)_{\mathrm{I}} = \left(m\frac{du}{d\nu}\right)_{\mathrm{O}} - \left(m\frac{du}{d\nu}\right)_{\mathrm{I}}$$

and so on.

Now, using Lamé's notation for the elastic constants,

$$l\mathrm{P} + m\mathrm{U} + n\mathrm{T} = l\left(\lambda\Delta + 2\mu\frac{du}{dx}\right) + m\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) + n\mu\left(\frac{du}{dz} + \frac{dw}{dx}\right),$$

$$\text{where} \qquad \Delta \equiv \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}.$$

The above surface conditions may therefore be written

$$l(\lambda+\mu)\left(l\frac{du}{d\nu} + m\frac{dv}{d\nu} + n\frac{dw}{d\nu}\right) + \mu\frac{du}{d\nu} = \text{same in O and I}$$

$$m(\lambda+\mu)\left(\qquad \text{same} \qquad\right) + \mu\frac{dv}{d\nu} = \text{same in O and I}$$

$$n(\lambda+\mu)\left(\qquad \text{same} \qquad\right) + \mu\frac{dw}{d\nu} = \text{same in O and I.}$$

Multiplying these equations by l, m, n and adding we find

$$l\frac{du}{d\nu} + m\frac{dv}{d\nu} + n\frac{dw}{d\nu} = \text{same in O and I};$$

and then, from each of the equations in turn, we find that

$$\frac{du}{d\nu}, \quad \frac{dv}{d\nu}, \quad \frac{dw}{d\nu} \quad \text{are continuous.}$$

Hence all the first derivatives of u, v, w are continuous.

5. In a homogeneous isotropic solid subjected to applied force

whose components at a point (x, y, z) have any finite values X, Y, Z per unit volume, the displacements u, v, w at (x, y, z) satisfy the equations

$$\mu\nabla^2 u + (\lambda+\mu)\frac{d\Delta}{dx} + X = 0$$

$$\mu\nabla^2 v + (\lambda+\mu)\frac{d\Delta}{dy} + Y = 0$$

$$\mu\nabla^2 w + (\lambda+\mu)\frac{d\Delta}{dz} + Z = 0$$

and we have just seen that whether X, Y, Z be continuous or not, u, v, w and their first derivatives are finite and continuous throughout.

Taking now any given state of strain, and any surface S within the solid, we may, without altering the state of strain outside S, replace u, v, w within S by any other functions u', v', w', which, with their first derivatives, are finite and continuous, provided

$$u', \quad v', \quad w', \quad \frac{du'}{d\nu}, \quad \frac{dv'}{d\nu}, \quad \frac{dw'}{d\nu}$$

have at the surface S the same values as

$$u, \quad v, \quad w \quad \frac{du}{d\nu}, \quad \frac{dv}{d\nu}, \quad \frac{dw}{d\nu} \quad \text{respectively.}$$

The new state of strain will require, within S, bodily forces given explicitly by the above equations.

As an interesting special case, u, v, w may be zero at all points outside S, provided u', v', w', $\frac{du'}{d\nu}$, $\frac{dv'}{d\nu}$, $\frac{dw'}{d\nu}$ are all zero at S. Thus we can find any number of systems of force within S which will produce absolutely no effect outside S.

6. Again, the displacements

$$U_1 = -\frac{d^2r}{dx^2} + \frac{\lambda+2\mu}{\lambda+\mu}\,\frac{2}{r}$$

$$V_1 = -\frac{d^2r}{dx\,dy}$$

$$W_1 = -\frac{d^2r}{dx\,dz}$$

where r is the distance from (x, y, z) to a fixed point (x', y', z'), are at once seen $\left(\text{observing that } \nabla^2 r = \frac{2}{r}\right)$ to satisfy the equations of equilibrium under no forces at all points except (x', y', z'), at which they become infinite.

Taking a sphere of radius a about $(x'y'z')$ as centre, we may find, in the way just explained, a system of force within this sphere which will produce the above displacements at all external points.

Writing ξ, η, ζ for the coordinates of (x, y, z) relative to (x', y', z'), the values of U_1, V_1, W_1 at the surface of the sphere are

$$
\begin{aligned}
U_1 &= \frac{\xi^2}{a^3} + \frac{\lambda + 3\mu}{\lambda + \mu}\,\frac{1}{a} \\
V_1 &= \frac{\xi\eta}{a^3} \\
W_1 &= \frac{\xi\zeta}{a^3} \quad .
\end{aligned}
$$

Retaining the displacements U_1, V_1, W_1 outside the sphere, take the displacements within

$$
\begin{aligned}
U_0 &= \frac{\xi^2}{a^3} + \frac{\lambda + 3\mu}{\lambda + \mu}\,\frac{1}{a} + (r^2 - a^2)U \\
V_0 &= \frac{\xi\eta}{a^3} \qquad\qquad\qquad + (r^2 - a^2)V \\
W_0 &= \frac{\xi\zeta}{a^3} \qquad\qquad\qquad + (r^2 - a^2)W.
\end{aligned}
$$

Then U_0, V_0, W_0 are equal to U_1, V_1, W_1 at the surface; in order that $\frac{dU_0}{dr}$, $\frac{dV_0}{dr}$, $\frac{dW_0}{dr}$ should be equal to $\frac{dU_1}{dr}$, $\frac{dV_1}{dr}$, $\frac{dW_1}{dr}$ we must have at the surface

$$
\begin{aligned}
U &= -\tfrac{3}{2}\frac{\xi^2}{a^5} - \frac{\lambda + 3\mu}{\lambda + \mu}\,\frac{1}{2a^3} \quad . \\
V &= -\tfrac{3}{2}\frac{\xi\eta}{a^5} \\
W &= -\tfrac{3}{2}\frac{\xi\zeta}{a^5} \quad .
\end{aligned}
$$

It is sufficient for our purpose to take U, V, W as having these values throughout the sphere.

The components X_0, Y_0, Z_0 of the force within the sphere can be readily calculated. They are found to be of the form

$$X_0 = \mu\left(\frac{A}{a^3} + \frac{B\xi^2}{a^5} + \frac{Cr^2}{a^5}\right)$$

$$Y_0 = \mu \,.\quad \frac{D\xi\eta}{a^5}$$

$$Z_0 = \mu \,.\quad \frac{D\xi\zeta}{a^5},$$

where A, B, C, D are *numbers*.

The resultant of this system of forces is a single force parallel to the axis of x, and passing through $(x'y'z')$; its magnitude is

$$\frac{8\pi\mu(\lambda + 2\mu)}{\lambda + \mu};$$

the reciprocal of this we shall denote by M.

7. Take now for the displacements of a solid

$$u_1 = \mathrm{M}\,\mathrm{U}_1 = \mathrm{M}\left(-\frac{d^2r}{dx^2} + \frac{\lambda + 2\mu}{\lambda + \mu}\,\frac{2}{r}\right)$$

$$v_1 = \mathrm{M}\,\mathrm{V}_1 = -\mathrm{M}\frac{d^2r}{dx\,dy}$$

$$w_1 = \mathrm{M}\mathrm{W}_1 = -\mathrm{M}\frac{d^2r}{dx\,dz}$$

at all points external to the sphere of radius a about $(x'y'z')$; and MU_0, MV_0, MW_0 at all internal points.

This system is maintained by the system of forces MX_0, MY_0, MZ_0 within the sphere, the resultant of which is a unit force parallel to Ox, through the point $(x'y'z')$; along with, if the solid is bounded externally by a surface S completely enclosing the sphere, surface tractions on S which we shall denote by F_1, G_1, H_1, immediately calculable when the surface S is known.

With a view to applying Betti's theorem, take along with this system a second system in which the external surface S is held fixed by tractions F, G, H, and in which the displacements and components of force per unit volume at any point (xyz) are (u, v, w); (X, Y, Z).

Betti's theorem gives

$$\iiint(Xu_1+Yv_1+Zw_1)dx\,dy\,dz+\iiint M(XU_0+YV_0+ZW_0)d\xi\,d\eta\,d\zeta$$
$$+\iint(Fu_1+Gv_1+Hw_1)dS=\iiint M(X_0u+Y_0v+Z_0w)d\xi\,d\eta\,d\zeta,$$

the first volume integral being taken through the space between S and the sphere, and the other two through the sphere.

The integral $\iiint(Xu_1+Yv_1+Zw_1)d\xi\,d\eta\,d\zeta$ taken through the sphere, where u_1, v_1, w_1 are supposed to retain their values as given above right up to (x', y', z'), is clearly finite, since u_1, v_1, w_1 are of the order $\frac{1}{r}$ in the neighbourhood of $(x'y'z')$; adding this integral to both sides of the above equation, and transposing the second integral, we have

$$\iiint(Xu_1+Yv_1+Zw_1)dx\,dy\,dz+\iint(Fu_1+Gv_1+Hw_1)dS$$
$$=\iiint(Xu_1+Yv_1+Zw_1)d\xi\,d\eta\,d\zeta+\iiint M(X_0u+Y_0v+Z_0w)d\xi\,d\eta\,d\zeta$$
$$-\iiint M(XU_0+YV_0+ZW_0)d\xi\,d\eta\,d\zeta,$$

the first volume integral being now taken over the whole solid.

The left hand member is independent of the radius a; hence so also must the right. To find the value of the latter, suppose a to be indefinitely diminished. The first and third of the integrals vanish in the limit; the second, from the form of X_0, Y_0, Z_0 as given in § 6, becomes

$$u'\times\ \text{Limit of}\iiint MX_0d\xi\,d\eta\,d\zeta+\ \text{two similar terms}$$

i.e. simply u', since the forces MX_0, MY_0, MZ_0 have for resultant a unit force parallel to Ox;
where u' is the value of u at $(x'y'z')$.

Hence $u'=\iiint Xu_1+Yv_1+Zw_1)dx\,dy\,dz+\iint(Fu_1+Gv_1+Hw_1)dS.$

If now we suppose the solid to extend to infinity, and to be

subjected to force, continuous or discontinuous, throughout its whole extent, we have for the displacement u' at any point $(x'y'z')$, (the solid being fixed at infinity)

$$u' = \iiint (\mathrm{X}u_1 + \mathrm{Y}v_1 + \mathrm{Z}w_1)dx\,dy\,dz,$$

provided X, Y, Z are such that the integral $\iint(\mathrm{F}u_1 + \mathrm{G}v_1 + \mathrm{H}w_1)d\mathrm{S}$,

taken over the surface at an infinite distance, vanishes;

and provided that the integral $\iiint(\mathrm{X}u_1 + \mathrm{Y}v_1 + \mathrm{Z}w_1)dx\,dy\,dz$ is finite.

The values of v', w' may be written down from symmetry.

8. Apart from the preceding application, the process of last section shows that we may apply Betti's theorem to the system u_1, v_1, w_1, supposed for mathematical purposes to extend right up to $(x'y'z')$, and that in calculating the work expressions, we must suppose in the system u_1, v_1, w_1 a unit force to exist at $(x'y'z')$.

We therefore, in the remainder of the paper, treat $u_1v_1w_1$, without reserve, as the displacements due to a unit force parallel to Ox at $(x'y'z')$.

For a unit force at $(x'y'z')$, parallel to Oy, the displacements are

$$\begin{aligned} u_2 &\equiv -\mathrm{M}\frac{d^2r}{dy\,dx} \\ v_2 &\equiv -\mathrm{M}\frac{d^2r}{dy^2} + \frac{\lambda+2\mu}{\lambda+\mu}\frac{2\mathrm{M}}{r} \\ w_2 &\equiv -\mathrm{M}\frac{d^2r}{dy\,dz} \qquad ; \end{aligned}$$

and for a unit force parallel to Oz

$$\begin{aligned} u_3 &\equiv -\mathrm{M}\frac{d^2r}{dy\,dx} \\ v_3 &\equiv -\mathrm{M}\frac{d^2r}{dz\,dy} \\ w_3 &\equiv -\mathrm{M}\frac{d^2r}{dz^2} + \frac{\lambda+2\mu}{\lambda+\mu}\frac{2\mathrm{M}}{r}. \end{aligned}$$

The symbols u_1, v_1, w_1; u_2, v_2, w_2; u_3, v_3, w_3 we shall use throughout as they are now defined; further, in the case when the solid is bounded by a surface S at a finite distance, the necessary tractions on S will be denoted by

$$F_1,\ G_1,\ H_1;\quad F_2,\ G_2,\ H_2;\quad F_3,\ G_3,\ H_3.$$

9. From these three fundamental solutions the solutions used by Betti, referred to in § 1, may be derived as follows:

(*a*) Take a force Q parallel to Oy at $(x'y'z')$, and a force $-Q$ parallel to Oy at $(x',\ y',\ z'+h)$.

The displacements due to the former are Qu_2, Qv_2, Qw_2; and to the latter $-Qu_2'$, $-Qv_2'$, $-Qw_2'$, say.

Let Q be increased and h diminished indefinitely, so that Qh remains finite and equal to $\frac{1}{2}$.

The x - displacement due to the combination is $Q(u_2-u_2')$, or

$$-Qh\frac{u_2'-u_2}{h},\quad \text{which in the limit} \;=-\tfrac{1}{2}\frac{du_2}{dz'}\quad \text{or}\quad \tfrac{1}{2}\frac{du_2}{dz}.$$

The displacements are, therefore, $\frac{1}{2}\frac{du_2}{dz}$, $\frac{1}{2}\frac{dv_2}{dz}$, $\frac{1}{2}\frac{dw_2}{dz}$.

The resultant of the forces applied to the element at $(x'y'z')$ is a couple of moment $\frac{1}{2}$ in the yz plane;

the work done by these forces acting through any system of displacements u, v, w is $-\frac{1}{2}\left(\frac{dv}{dz}\right)'$, the value of $-\frac{1}{2}\frac{dv}{dz}$ at $(x'y'z')$.

Again, if we take a force $-R$ parallel to Oz at $(x'y'z')$ and a force $+R$ parallel to Oz at $(x',\ y'+g,\ z')$, and proceed to the limit as before, keeping $Rg=\frac{1}{2}$,

the displacements will be $-\frac{1}{2}\frac{du_3}{dy}$, $-\frac{1}{2}\frac{dv_3}{dy}$, $-\frac{1}{2}\frac{dw_3}{dz}$;

the resultant of the forces on the element is again a couple of moment $\frac{1}{2}$ in the yz plane;

the work done by these forces acting through u, v, w is $\frac{1}{2}\left(\frac{dw}{dy}\right)'$.

Taking these two doublets together, the displacements are

$$u_4 = \tfrac{1}{2}\left(\frac{du_2}{dz} - \frac{du_3}{dy}\right) = 0$$

$$v_4 = \tfrac{1}{2}\left(\frac{dv_2}{dz} - \frac{dv_3}{dy}\right) = \frac{1}{8\pi\mu}\frac{d}{dz}\left(\frac{1}{r}\right)$$

$$w_4 = \tfrac{1}{2}\left(\frac{dw_2}{dz} - \frac{dw_3}{dy}\right) = -\frac{1}{8\pi\mu}\frac{d}{dy}\left(\frac{1}{r}\right);$$

the resultant of the forces at $(x'y'z')$ is a unit couple in the yz plane; the work done on any displacements u, v, w is $\tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right)'$ $= \omega_1'$, the value at $(x'y'z')$ of the x-rotation in the system u, v, w.

Similar results hold for the similarly derived systems

$$u_5 = -\frac{1}{8\pi\mu}\frac{d}{dz}\left(\frac{1}{r}\right) \qquad u_6 = \frac{1}{8\pi\mu}\frac{d}{dy}\left(\frac{1}{r}\right)$$

$$v_5 = 0 \qquad ; \qquad v_6 = -\frac{1}{8\pi\mu}\frac{d}{dx}\left(\frac{1}{r}\right)$$

$$w_5 = \frac{1}{8\pi\mu}\frac{d}{dx}\left(\frac{1}{r}\right) \qquad w_6 = 0$$

The tractions on the surface S required to maintain these we denote by F_4, G_4, H_4; F_5, G_5, H_5; F_6, G_6, H_6.

It is to be noticed that the resultant of the tractions F_4, G_4, H_4 on S must be a negative unit couple in the yz plane, and similarly with the others.

(*b*) Take next the following system of applied forces:

parallel to Ox, $-P$ at $(x'y'z')$ and $+P$ at $(x'+f, y', z')$;

parallel to Oy, $-Q$ at $(x'y'z')$ and $+Q$ at $(x', y'+g, z')$;

parallel to Oz, $-R$ at $(x'y'z')$ and $+R$ at $(x', y', z'+h)$.

Let P, Q, R increase, and f, g, h diminish indefinitely so that

$$Pf = Qg = Rh = 1.$$

Then ultimately the displacements are

$$u_7 = -\left(\frac{du_1}{dx}+\frac{du_2}{dy}+\frac{du_3}{dz}\right) = -\frac{1}{4\pi(\lambda+2\mu)}\frac{d}{dx}\left(\frac{1}{r}\right)$$

$$v_7 = -\left(\frac{dv_1}{dx}+\frac{dv_2}{dy}+\frac{dv_3}{dz}\right) = -\frac{1}{4\pi(\lambda+2\mu)}\frac{d}{dy}\left(\frac{1}{r}\right);$$

$$w_7 = -\left(\frac{dw_1}{dx}+\frac{dw_2}{dy}+\frac{dw_3}{dz}\right) = -\frac{1}{4\pi(\lambda+2\mu)}\frac{d}{dz}\left(\frac{1}{r}\right)$$

the applied forces are in equilibrium;
the work done on any displacements u, v, w is $\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)'$,
the value at $(x'y'z')$ of the delatation Δ in the system u, v, w.
The tractions on S required to maintain u_7, v_7, w_7 we denote by F_7, G_7, H_7.

10. Consider now any system of displacements u, v, w of the solid bounded by the surface S, produced by surface tractions alone, F, G, H.

Apply Betti's Theorem to this system taken along with each of the above systems u_1, v_1, w_1, etc., in turn: we find

$$u' = \iint(Fu_1 + Gv_1 + Hw_1)dS - \iint(F_1u + G_1v + H_1w)dS \quad - \quad (1)$$

$$v' = \iint(Fu_2 + Gv_2 + Hw_2)dS - \iint(F_2u + G_2v + H_2w)dS \quad - \quad (2)$$

$$w' = \iint(Fu_3 + Gv_3 + Hw_3)dS - \iint(F_3u + G_3v + H_3w)dS \quad - \quad (3)$$

$$\omega_1' = \iint(Fu_4 + Gv_4 + Hw_4)dS - \iint(F_4u + G_4v + H_4w)dS \quad - \quad (4)$$

with two similar equations - - (5) and (6)

$$\Delta' = \iint(Fu_7 + Gv_7 + Hw_7)dS - \iint(F_7u + G_7v + H_7w)dS \quad - \quad (7)$$

The equations from (4) to (7) are those on which Betti founds his method of solving the equations of equilibrium under given surface conditions; we propose to develop a similar method from the more fundamental relations (1), (2), (3).

It may be noticed that if *both* surface displacements and surface tractions were given, these relations would give explicitly the values of the displacements at any internal point $(x'y'z')$.

11. Suppose, in the first place, that the displacements u, v, w are given at every point of S; it is required to find u, v, w at $(x'y'z')$.

Let F_1', G_1', H_1' be the tractions on S required to give surface displacements equal to u_1, v_1, w_1, the solid being free from internal applied force; and in this case let u_1', v_1', w_1' be the internal displacements.

Applying Betti's theorem to the systems u, v, w and u_1', v_1', w_1' we have, since at the surface u_1', v'_1, w'_1 are equal to u_1, v_1, w_1

$$\iint (Fu_1 + Gv_1 + Hw_1)dS - \iint (F_1'u + G_1'v + H_1'w)dS = 0.$$

Combining this with (1) of § 10, we get

$$u' = \iint \{(F_1' - F_1)u + (G_1' - G_1)v + (H_1' - H_1)w\} dS.$$

Hence if F_1', G_1', H_1' can be found, the value of u at any internal point $(x'y'z')$ is determined.

We observe that the tractions $F_1 - F_1'$, $G_1 - G_1'$, $H_1 - H_1'$, acting along with the unit force at $(x'y'z')$ give zero displacements at the surface S.

Similar relations may at once be written down for v', $w,'$ for the rotations, and for the dilatation at $(x'y'z')$.

12. The process just explained for solving the problem of given surface displacements, requires us to find surface tractions which, acting along with a given internal source of strain, will hold the surface *fixed*; the corresponding process, when the surface tractions are given, is more complicated, since the surface cannot be *free*, unless the internal applied forces are in equilibrium.

For this application we shall therefore suppose applied to the solid, in addition to a unit force at the point whose displacement is to be found, a balancing force and couples acting on the element at some selected point of the solid, taken as origin of coordinates; in a sphere, for example, this point would naturally be taken at the

centre; in a solid bounded by a plane, or by two parallel planes, it would be convenient to take it at an infinite distance.

Denote by capital letters U, V, W the displacements at any point of the solid arising from a source at the origin, of any of the types already dealt with, and retain suffixes to denote the type of the source. For example, the displacements due to a unit force parallel to Ox applied at the origin denote by $U_1V_1W_1$; the displacements due to a source at O of the $(u_4v_4w_4)$ type, denote by $U_4V_4W_4$.

A unit force parallel to Ox at $(x'y'z')$ is balanced by a negative unit force parallel to Ox at O, with a couple $-z'$ about Oy, and a couple $+y'$ about Oz.

Hence the displacements

$$\begin{aligned} u_8 &= u_1 - U_1 - z'U_5 + y'U_6 \\ v_8 &= v_1 - V_1 - z'V_5 + y'V_6 \\ w_8 &= w_1 - W_1 - z'W_5 + y'W_6 \end{aligned}$$

will give rise to tractions F_8, G_8, H_8 on the surface, which are in equilibrium.

Apply Betti's Theorem to the systems u, v, w and u_8, v_8, w_8. By the results of § 10, we have

$$\iint (Fu_8 + Gv_8 + Hw_8)dS - \iint (F_8u + G_8v + H_8w)dS \\ = u' - u - z'\omega_2 + y'\omega_3, \tag{8}$$

where u' is the x-displacement at $(x'y'z')$, and u, ω_2 ω_3 are the values at O of the x-displacement and the y and z rotations, all in the system u, v, w.

Let u_8', v_8', w_8' be the displacements due to the equilibrating system F_8, G_8, H_8 with no internal force.

By Betti's Theorem

$$\iint (Fu_8' + Gv_8' + Hw_8')dS - \iint (F_8u + G_8v + H_8w)dS = 0.$$

Combining this with (8) we have

$$u' - u - z'\omega_2 + y'\omega_3 = \iint \{F(u_8 - u_8') + G(v_8 - v_8') + H(w_8 - w_8')\}dS$$

Clearly $u_8 - u_8'$, $v_8 - v_8'$, $w_8 - w_8'$ are the displacements due to

the compound system of sources giving rise in an infinite solid to u_8, v_8, w_8, in the case when the surface is *free*.

The result illustrates the known principle that when the surface tractions are given, the displacements are indeterminate to the extent of an arbitrary displacement of the solid as a rigid body.

The solution can be made determinate if it is arranged that in all cases of given surface tractions, the position of the solid shall be so adjusted that displacements and rotations vanish at some one assigned point.

If this point be the point O above, we have simply

$$u' = \iint \{F(u_8 - u_8') + G(v_8 - v_8') + H(w_8 - w_8')\} dS.$$

Similar relations hold, of course, for v', w'.

13. Since the internal force producing the system $u_4 v_4 w_4$ reduces to a unit couple in the yz plane on the element at $(x'y'z')$, it follows that the displacements

$$\begin{aligned} u_9 &= u_4 - U_4 \\ v_9 &= v_4 - V_4 \\ w_9 &= w_4 - W_4 \end{aligned}$$

will require on the surface S tractions $F_9 G_9 H_9$ which are in equilibrium.

Apply Betti's Theorem to u, v, w and $u_9 v_9 w_9$.

$$\text{Hence} \iint (Fu_9 + Gv_9 + Hw_9) dS - \iint (F_9 u + G_9 v + H_9 w) dS \\ = \omega_1' - \omega_1 \qquad (9)$$

ω_1', ω_1 being the x-rotations in the system u, v, w at $(x'y'z')$ and O respectively.

Let u_9', v_9', w_9' be the displacements due to the equilibrating tractions F_9, G_9, H_9 with no internal force.

Hence

$$\iint (Fu_9' + Gv_9' + Hw_9') dS - \iint (F_9 u + G_9 v + H_9 w) dS = 0.$$

Combining with (9), we have

$$\omega_1' - \omega_1 = \iint \{F(u_9 - u_9') + G(v_9 - v_9') + H(w_9 - w_9')\} dS.$$

Similarly for the y and z rotations.

These formulae for the rotations, and the corresponding formulae

for the dilation, may of course be deduced directly from the equations of §§11, 12, by differentiation with respect to $(x'y'z')$. It may be noted that Betti obtains in place of (9) a formula giving ω_1' simply without ω_1, but he overlooks the remark at the beginning of §12. The error is not corrected in Dr Love's text-book.

14. The general problem of an elastic solid with given bodily forces and given surface tractions or displacements, may be divided into two parts; in the first, we suppose the bodily force to be null, and the surface tractions or displacements to have their given values; in the second, we suppose the bodily forces to exist, but the surface to be free or fixed. The first part we have dealt with; suppose now that any system of bodily forces X, Y, Z per unit volume exists in the solid, and that

(*a*) the surface is fixed.

Referring to §11, the displacements when a unit x-force is applied at $(x'y'z')$, and the surface is fixed, are

$$u_1 - u_1', \qquad v_1 - v_1', \qquad w_1 - w_1'.$$

Take this system with the system u, v, w due to X, Y, Z. Betti's Theorem gives for u' the x-displacement at $(x'y'z')$ in the system u, v, w.

$$u' = \iiint \{\mathrm{X}(u_1 - u_1') + \mathrm{Y}(v_1 - v_1') + \mathrm{Z}(w_1 - w_1')\} dx\, dy\, dz$$

the integral being taken over the whole solid.

Similarly with v', w'.

(*b*) The surface is free.

Referring to §12, the displacements due to a unit x-force at $(x'y'z')$ with balancing sources at O, when the surface is free, are

$$u_8 - u_8', \qquad v_8 - v_8', \qquad w_8 - w_8'.$$

Taking this system with the system u, v, w due to X, Y, Z when the surface is free, we have

$$u' - u - z'\omega_2 + y'\omega_3 = \iiint \{\mathrm{X}(u_8 - u_8') + \mathrm{Y}(v_8 - v_8') + \mathrm{Z}(w_8 - w_8')\} dxdydz.$$

When, in the general problem, the systems of bodily force and of surface traction are not separately in equilibrium, a slight modification of the process, such as is exemplified in §12, will be required, but this need hardly be dealt with at length.

[*The following Paper was read at the Fourth Meeting, 11th February* 1898.]

The Treatment of Proportion in Elementary Geometry.

By Professor GIBSON.

I do not think any apology is needed for asking the Society to consider the treatment of Proportion in Elementary Geometry. Although the fifth book of Euclid's *Elements* appears in all editions of Euclid, I know of no school or college where it is read; I know of no examination for which it is prescribed, and I have never seen an examination paper which contained a question based upon it, except in regard to its definitions. Indeed I believe it is not unfair to say that even among teachers themselves a thorough knowledge of Euclid's fifth book is very rare.

In a country where respect for Euclidian methods borders on superstition, this is surely a striking state of matters, and there can, I think, be little doubt that most teachers are far from satisfied with the practice now usually adopted of getting over the difficulties of the Euclidian theory. The usual practice, so far as I can determine, is to go over the definitions of the fifth book, and to prove the first and the thirty-third propositions of the sixth book in Euclid's manner, but to adopt the arithmetic or algebraic proofs of the theorems of the fifth book, some attempt being made to connect Euclid's definition of proportion with that given in Algebra. The version of Book V. in the text-book of the A.I.G.T. does not seem to have fared better than Euclid's own; at any rate, I have not met any teacher who adopts its proofs of the theorems required in the application of proportion to geometry.

The difficulty of the situation is increased by the fact that Euclid's treatment of proportion is *in itself* admirable, and while he did not, I think, give the full development of the conception of a ratio, as distinguished from a proportion, of which his method is capable, all recent researches into the representation of continuous magnitude by number have only put in a stronger light the intrinsic excellence of his method. It is therefore from no disparagement of the scientific value of the Euclidian theory of proportion that I urge

a different method of dealing with it; but it is quite impossible to overlook the fact that in elementary teaching Euclid's method has broken down. The current practice of using only his definitions seems to me a very unsatisfactory makeshift, especially because I believe that we have ready to our hand an equally rigorous method, that is free from the chief difficulties of Euclid's. If we had to teach pupils whose intellect was sufficiently matured to deal intelligently with abstract conceptions, then there would be no great need for abandoning the Euclidian theory; but we can never expect to have such pupils.

In teaching Euclid's definitions of ratio and proportion, the prime difficulty that I have found has been to connect in a satisfactory way Euclid's definitions with those the pupil has been accustomed to use in arithmetic. And just at this point, I think, the unsatisfactoriness of Euclid's method for elementary teaching is most clearly seen. I suppose we may take it as axiomatic that we should, as far as possible, appeal to the conceptions already present in the pupil's mind and, it may be, already partially reduced to the definiteness required for mathematical work, and we should meet new difficulties, not so much by discarding the old conceptions as by developing them in such a way that the new form shall be seen to be but a generalisation of the old form. All through his mathematical training the pupil is guided on these lines; at a very early stage the primary conception of number as an integer is extended so as to include the conception of number as a fraction, and the usual operations on integers are carried over with the same names to the fractional number, though at first sight the names themselves seem often very unsuitable. In algebra this process is carried much farther, but at every stage the propriety of the extension is shown by a proof of the identity of certain fundamental elements common to all the stages.

Now in arithmetic "the ratio of one whole number to another is measured by the fraction which the one is of the other; and the ratio of one quantity to another is the ratio of the two whole numbers that express these quantities in terms of the same unit" (*E.M.S. Proc.*, VI., p. 98). It is certainly not easy to see the connection between ratio defined in this way and ratio as defined in Euclid's fifth book. In the text of Euclid there may be said to be two definitions, of which the first is "ratio is the (or a) relation of

two magnitudes of the same kind to one another in respect of quantuplicity." It may be remarked that this definition has a rather curious history. Barrow, a most learned and ingenious defender of the Euclidian theory, says, after devoting a lecture to the exposition and defence of the definition, that Euclid perhaps gave it only as "a prelude for method or ornament's sake to the more accurate definitions of *the same*, *a greater*, and *a less ratio* . . . that he might insinuate a certain general idea of ratio into the minds of the learners by this *metaphysical* definition; I say, *metaphysical*, for it is not properly mathematical, since it has no dependence upon it, nor is, or I believe can be, deduced in the Mathematics." (*Geom. Lect. XVIII.*) Simson says that he fully believes the definition is not Euclid's, but is the addition of some unskilful editor. On the other hand, it is this definition which has bulked most largely in recent discussions. Thus a reviewer writes, "we are inclined to think no treatise on geometrical proportion complete which does not give a thorough discussion of the theory of proportion based on *quantuplicity*." (*Math. Gaz.*, *April* 1896, p. 23).

But, passing over such contradictory estimates, it may be asked —Is the definition such as to make the general conception of ratio more definite and more suitable for mathematical purposes than it is without the definition? Or again, take the other definition, namely "magnitudes are said to have a ratio to one another when they can, being multiplied, exceed the one the other," which is usually and correctly interpreted to mean that magnitudes can have a ratio to each other only when they are of the same kind. Does the former definition give any clearer idea than the latter? Certainly not, until the word *quantuplicity* is defined, and even when this has been done, there seems a good deal to accomplish. Now what is the meaning of the Greek word πηλικότης translated *quantuplicity*? Barrow translates it by *quantity*, and objects to the translation *tantuplicity*; he quotes with approval a Greek scholiast who says he thinks Euclid designedly put *according to quantity*, rather than *according to quality*, because all ratios are not capable of being expressed by number. Again, to elucidate the meaning of the word, the *Syllabus* of the *A.I.G.T.* states that "the quantuplicity of A with respect to B may be estimated by examining how the multiples of A are distributed among the multiples of B when

both are arranged in ascending order of magnitude, and the series of multiples continued without limit." It would seem as if we were defining one term by another which is certainly not simpler, as it should be, according to all the rules of logic.

But translate the word as we may, what is to be understood by the definition? Take the simple case when A is five-thirds of B; what is the ratio of A to B according to the definition? One interpreter of this order of ideas says "that relation in virtue of which A is a fraction of B—a fraction being defined as the ratio of two numbers—is called the ratio of A to B when they are commensurable." (*Nixon*, *Euc. Rev.*, p. 223, 1st Ed.) This is surely an extraordinary statement; notice that the ratio of two numbers is not defined as a fraction, but *the fraction* is defined as the *ratio* of two numbers, so that we apparently are defining the fundamental conception of a ratio by a ratio itself. Suppose, however, that we say the relation that we are considering between A and B is the same as the relation between the numbers 5 and 3, we must determine which of the many relations between 5 and 3 is to be fixed upon as the relation to be denominated their ratio. It is assumed to be neither their sum, their difference, nor their product; the next simplest is the quotient of 5 by 3, and this relation is expressed by the fraction $\frac{5}{3}$. Does the definition then mean that when A is equal to five-thirds of B, the ratio of A to B is that relation between the numbers 5 and 3 which is expressed by taking the quotient of 5 by 3? This statement is, I should say, intolerably prolix, but unless this be the meaning of the definition, I am quite at a loss to say what it means. And I think it is just possible to render the Greek words, without putting any strain upon them, so to bring out this meaning more fully, thus "ratio is a sort of quotient-relation between two homogeneous magnitudes." If it were worth while to go into the matter, I think it might be shown that Eutocius, the commentator on Archimedes, explains the word πηλικότης to mean the same as our word "quotient"; the translation suggested also gives the natural signification to the particle ποιά.

In the case of commensurables, then, that is when $n\text{A} = m\text{B}$, m, n being integers, the ratio of A to B is the quotient relation between m and n; but when A and B are incommensurable, the above method of expounding the definition is at fault, since there are no

integers such that $nA = mB$. What, then, is to be understood by *quantuplicity?* If we examine the infinite sequence of multiples of A and B, we may find for any multiple of A, say nA, two multiples of B, say mB and $(m+1)B$ between which the multiple nA lies. But the ratio of A to B is neither the quotient relation between m and n, nor that between $m+1$ and n, so that we do not know what the ratio is; there is, in fact, *no quotient relation* between A and B so long as we are restricted to integers. The word *quantuplicity* is not a whit simpler than *ratio*, and the statement quoted above from the *Syllabus* of the manner of estimating the quantuplicity of A with respect to B seems to me to assume that quantuplicity is a relation between numbers. Now so long as number means *rational number*, there is no relation between the incommensurable magnitudes A and B that can be said to be the same as the relation between two numbers. By a stretch of language the relation may be said to be greater than the relation between m and n, but less than that between $m+1$ and n; but even so, it does not tell us what the relation *is*.

It seems to me, then, that it is impossible to get out of the definition any precise meaning for the case of incommensurable magnitudes. Besides, I quite agree with Barrow that the definition is *metaphysical*; it is only by a considerable strain on the meaning of the word *quantuplicity* that we can get a precise meaning even for the simple case of commensurable magnitudes. And when all is said and done, the definition is not of the slightest use; nothing in the subsequent development of Euclid's theory depends on it, nor does it appear in the definition of proportion. I think it is a complete mistake, when treating ratio in Euclid's manner, to define it in any other way than as "a relation between like magnitudes." What precisely that relation is can only appear after the definition of equality of ratios; when the test of equality has been given, there remains the task of assigning the laws of operation to which it is subject. When that task is completed, it is seen that a ratio possesses in every respect the properties of number, meaning by number, rational and irrational number. No doubt Euclid does not carry its developments to its farthest limits; but it is to be observed that Euclid does not speak of ratios as we do. He does not say "two ratios are equal," they are only "the same," and he considers it necessary *to prove* that ratios obey the general test of equality.

This course was necessary to him from the abstract way in which he defines a ratio, and it seems to me to support Barrow's contention that he did not consider a ratio to be a quantity at all. It belonged to the category of "relations" and not to that of "quantities"; in the language of Barrow, "ratio is not a genus or kind of quantity, nor anything subject to quantity, or anywhere properly attributed to quantity directly by itself, but agrees no otherwise with it, than by a *catachresis* or *metonymy*." (*Lect.* XX., p. 368.) To the obvious objection that the mode of expression, "one ratio is equal to, greater than or less than another ratio" suggests that ratio is a quantity. Barrow replies, the expression really means that when the ratios are reduced to having a common consequent, the *antecedent* of the one is equal to, greater than or less than the *antecedent* of the other. (p. 377.)

It is quite unnecessary to follow Barrow's defence of Euclid any further; but I have discussed Euclid's definition of ratio at considerable length in order to show that when we approach the consideration of ratio from his standpoint, we have to put out of sight altogether the arithmetical conception of a ratio. Whether the quantuplicity definition be Euclid's or not, its absence from Book V. would not render the slightest change necessary in any part of the Book.

But, further, there is a vagueness that is most undesirable in elementary teaching in speaking of a ratio as "a relation." There may perhaps be nothing wrong in saying that the ratio of two magnitudes is a certain relation between them, but all through mathematics it is the *quantitative value* of the ratio that is really meant by the word. Thus we define the sine of an angle as a ratio, and we say that sin 30° is $\frac{1}{2}$. In pure geometry this quantitative idea may not be so prominent, but even there it seems to me to be the radical idea; when we want to make the first proposition of Book VI. quite definite, we are accustomed to say, "if the base be doubled, the triangle is doubled; if the base be halved, the triangle is halved" and so on; it is the quantitative aspect of the relation that is of importance. Hence I contend that for elementary teaching it is essential that the numerical aspect of ratio should be insisted upon, both because that is the important element in mathematics and because it is directly in line with the ordinary use of the word in arithmetic and in common life.

Let us now consider the method usually adopted for proving the theorems in proportion required in the applications of proportion to geometry. Only the definitions of Book V. are supposed to be learned by the pupil; the proofs of the theorems are established algebraically, it being first shown that Euclid's test of proportionality leads to the algebraical test. The reason alleged for this procedure is that Euclid's test is applicable to all magnitudes, whether commensurable or incommensurable, while the algebraical is, in the usual phraseology, applicable "strictly speaking" to commensurable magnitudes alone. Now the only intelligible meaning I can give to this language is, that the ratio of two commensurable magnitudes is a number, or may be expressed by a number, while the ratio of two incommensurable magnitudes is not a number, and can not be expressed by a number. Inexpressibility as a number is, as I understand, the reason for adopting a definition of ratio in terms of quantuplicity.

It is to be borne in mind that Euclid's definition of proportion or of equality of ratios *is in his order of ideas not a theorem capable of proof;* it is a complete misapprehension of Euclid's position to say, as is sometimes done, that his definition of proportion is a theorem. Euclid does not define a ratio as a number, and his definition does not confer properties on it. His definition of equality of ratios is the first step in the process of endowing the abstract relation with definite properties. We are therefore not entitled to assume that a ratio is a quantity homogeneous with number and possessing the same laws of combination as numbers; this is the final and not the first stage in this theory. Seeing that he does not define a ratio as a quantity subject to the laws of algebraic operation and homogeneous with number, it is quite illegitimate to reason about a ratio as if it were an algebraic quantity until the proof has been given that this thing called a ratio, which by hypothesis is not a number, is actually subject to the operative laws of number. Euclid himself is perfectly consistent; he does not even assume that two ratios which are each equal to a third are equal to each other.

Now every method that I have seen of passing from Euclid's test of proportionality to the algebraic test assumes that a ratio is a quantity of the ordinary algebraic type. Thus take § 409 of Todhunter's *Algebra*, which is in substance identical with the

corresponding exposition of Barrow, though Barrow states and Todhunter does not state the really unsatisfactory point of the demonstration. Todhunter begins thus:—"Let a, b, c, d be four magnitudes which are proportional according to Euclid's definition: then shall $\frac{a}{b}=\frac{c}{d}$. For if $\frac{a}{b}$ be not equal to $\frac{c}{d}$, one must be greater than the other, and it will be possible to find some fraction which lies between them." Now what is to be understood by the symbols $\frac{a}{b}$, $\frac{c}{d}$? Are these fractions? If so, they are not ratios —unless we assume either that a ratio is a fraction or else that the proof has been previously given that Euclid's ratio may be treated as a fraction. Again a, b, c, d are expressly called *magnitudes*, and yet without the slightest explanation one magnitude is divided by another. Even if a, b are taken to represent not the magnitudes but their measures, they will be, when the magnitudes are incommensurable, irrational numbers, and the whole basis of the Euclidian theory is that only rational numbers are to be employed. Look at the demonstration any way we please, its validity depends on the assumption that a ratio is a quantity of the same nature as a number and subject to the same laws of operation.

But if we go on to § 410 we find that the equation $\frac{a}{b}=\frac{c}{d}$ is "strictly speaking" not an equation at all, for it is said "the algebraical definition is, strictly speaking, confined to commensurable quantities." I will return, in a moment, to the conception latent in this sentence.

It seems to me, then, that the usual method of passing from Euclid's definition to the algebraical, and of then establishing the theorems of Book V. algebraically is thoroughly unsound, and the labour involved in trying to make a pupil understand Euclid's definitions is worse than wasted. But I am prepared to go even farther than this. Suppose that the theorems of Books V. and VI. have been acquired from the text of Euclid and that the pupil goes on to the study of trigonometry. There the trigonometrical functions are defined as ratios, and in every text-book with which I am acquainted, except De Morgan's, Euclid's ratio is treated exactly like an algebraic quantity. Now Euclid himself never reached the position that a ratio is in every case a number, and it

was quite outside the range of Greek mathematics to investigate the laws of operation of mathematical symbols. At any rate, after all has been done that Euclid does in Books V. and VI., the problem still remains of co-ordinating ratios with numbers. Thus, consider what is implied in the statement "$\sin 45^\circ = \frac{1}{\sqrt{2}}$." By definition sin 45° is the ratio of the side of a square to its diagonal, and the fourth proposition of Book VI. is appealed to in order to show that the ratio is independent of the size of the square. But on what grounds is it stated that when the side is represented by 1 the diagonal may be represented by the symbol $\sqrt{2}$ and that the ratio is equal to $\frac{1}{\sqrt{2}}$. Euclid did not require to use such a symbol, but every application we make of geometry brings us face to face with the irrational number, and I contend that none of the standard text-books (for De Morgan's is now beyond the reach of most teachers, let alone their pupils) gives any reasonable exposition of the connection between Euclid's ratio and the irrational number. When the whole theory is based upon the supposed impossibility of representing continuous magnitude by number, it is surely necessary to say what is meant when continuous magnitudes are treated as done in trigonometry by the help of number. Some writers go an extreme length in denouncing the treatment of ratio from the numerical standpoint. Thus Nixon (*Euc. Rev.*, 1st Ed., p. 264) makes the statement:—"It is sometimes said that to *compound* ratios is the same as to *multiply* them. This, as a general statement, is quite wrong. The term "multiply" is an arithmetic term, and though applicable to the ratios of commensurable quantities, has no meaning in relation to the ratios of incommensurables." Yet the same writer in his excellent treatise on trigonometry, defines the trigonometrical functions as ratios and "multiplies" them, divides them and treats them in all respects as numbers. What is a pupil to think when he compares the statement in Book VI. with the treatment of ratios in the trigonometry?

The whole difficulty seems to me to lie in the conception of number; in Euclid's theory and in the usual expositions of it, number means rational number, and so long as the conception of number is thus restricted there is really no choice. It surely needs no argument to prove that a symbol which by hypothesis is not a

number, has no place in a number-equation until the laws according to which it may be combined with numbers have been investigated. If, as seems to be the case, De Morgan and his followers consider that numerical value can only be expressed by rational numbers, and that arithmetic and algebra are only concerned with such numbers, then Todhunter's proof referred to above really proves nothing at all.

If we are ever to have a theory of proportion that proceeds on other lines than those of Euclid's theory, it is essential to abandon the restriction in regard to number. At bottom, the passage from the rational to the irrational number is identical with that from the ratio of commensurable to the ratio of incommensurable magnitudes; but the difficulties in the one case seem to me much less than in the other. It is a curious study to compare the treatment of irrational numbers in our text-books with that of ratio in geometry; the difficulties are in great part the same, but there is no attempt to deal with irrational numbers in the thorough way Euclid treats ratio. So far as I can make out, the irrational number is really nothing more than a mere symbol; one constantly finds such language as this, "that $\sqrt{2}$ does not exist"; there is a perpetual confusion between $\sqrt{2}$ and rational approximations to it, as if we could approximate to a thing which does not exist!

In De Morgan's "Elements of Trigonometry" this attitude is seen very clearly. On p. 2 it is stated that when the sides of a right-angled triangle contain a, b, c of any linear unit the equation $aa+bb=cc$ does not exist arithmetically when any of the sides is incommensurable with the linear unit. He goes on to explain that the true interpretation of the equation is the proportion

$$aa+bb:1 \quad :: cc:1$$

where $a:1$, $b:1$, $c:1$ are symbols denoting the ratios of the sides to the linear unit, and this interpretation is based on the fully developed Euclidian theory of proportion. There is, no doubt, much that is excellent in De Morgan's exposition, but it seems to me that De Morgan bases the exposition on the conception that numerical value can only be expressed by rational number. What his exposition of Euclid's theory really proves is, that the symbols for ratios are subject to exactly the same laws of operation as the symbols of rational numbers, and perform the same function

in distinguishing all magnitudes that rational numbers do in distinguishing commensurable magnitudes. The equation

$$aa + bb = cc$$

exists just as much when a is the symbol for a ratio as when it is the symbol for a number; algebra is in this point of view a calculus of ratios. But there is really no need for this distinction between ratio and number, and as a matter of fact all through analysis the word number is used as equivalent to ratio. Once the fraction is admitted as a number, there is no reason for denying the name to the ratio of incommensurable magnitudes. The notion that numerical value can only be expressed by rational numbers seems to me quite untenable, and the use of such phrases as "arithmetical number" or "existing arithmetically" when the pupil has advanced so far in his studies as to apply algebra to trigonometry, fosters a totally false conception of the essential nature of a number-system. What is he to think when he comes to analytical geometry or the calculus if his number-system is restricted to rational numbers? The equation of a curve is not really an equation; how can a continuous function even be defined? We never meet in our text-books the statement that an independent variable is really a ratio and not a number at all; indeed, it would seem as if an equation to a curve or a differential equation were a mere misnomer. From the strict Euclidian point of view I cannot see that a ratio fails to satisfy any of the laws of algebra, and that it may not take its place, when once it has been fully developed, in any algebraic equation whatever; the whole of analysis would then be a calculus of ratios, most rigorously established, and "ratio" would be the equivalent of "real number" as that word is used in analysis.

I come back then to the original contention that the completely developed ratio of Euclid answers in every respect to the conception of number. For elementary teaching, however, I maintain that it is best to begin where Euclid ends, and before going into any complete theory of proportion the nature of the irrational number should be explained. In this way we follow the natural order; if magnitudes were commensurable there would, I suppose, be no question about treating ratio as a number, and seeing that in all parts of mathematics, not in geometry merely, the irrational number forces itself on our consideration, I think it is much better

to extend the notion of number and adapt ratio to that, than to set up a new theory of ratio which will end, if it is to be of any use outside pure geometry, in ranking ratio as a number.

Addendum.—I have omitted the rest of the paper as read before the Society; in the omitted portion an attempt was made to present the conception of the irrational number in a form suitable for school use, but as the essentials of the presentation are to be found in easily accessible text-books, it has been thought unncessary to retain the discussion. I think the Society, as a body, should make an effort to place the teaching of proportion on a more logical basis; it is not creditable that the present state of matters should continue. No doubt, the difficulties are considerable; but it would at least be better frankly to acknowledge these, and, if it be found impossible either to develop Euclid's conception to its natural completion or to provide a substitute in the form of a proper treatment of the irrational number, it would be less hurtful to confine all proofs to cases of commensurable magnitudes and to abandon the utterly illogical method so much in vogue of passing from the Euclidian to the algebraic definition of proportion. In any case, I think that in approaching the theory of proportion in geometry, attention should first be confined to commensurable magnitudes, and that the full theory should be taken up after the pupil has gained some familiarity with the processes involved.

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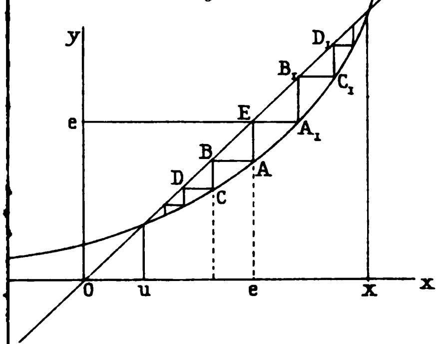

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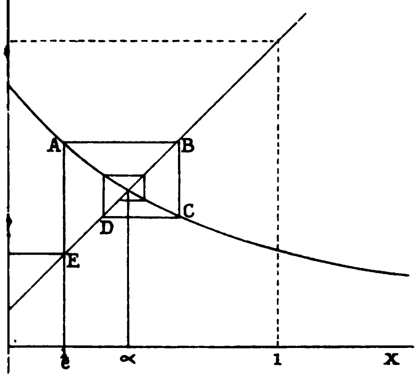

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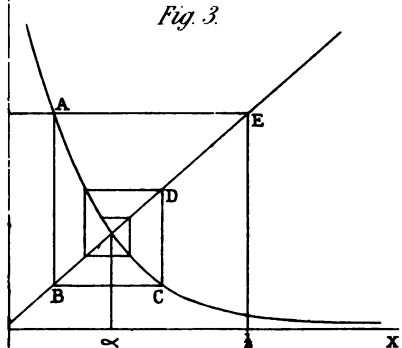

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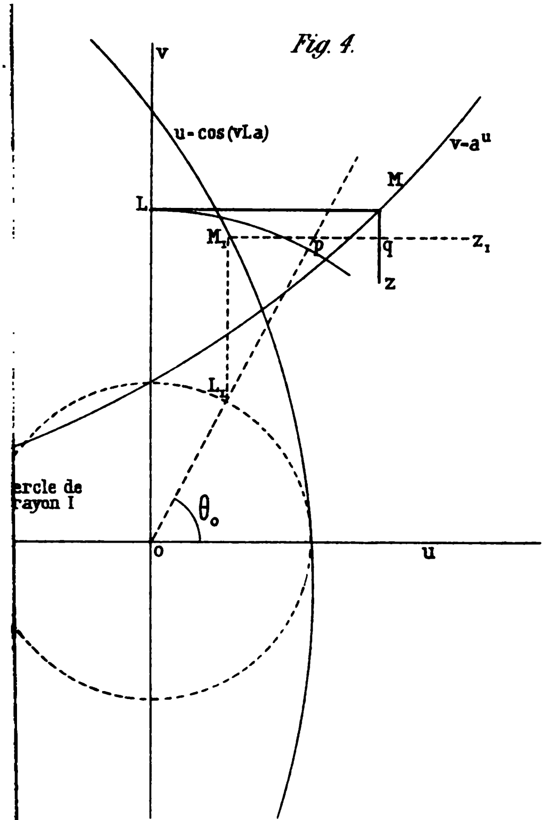

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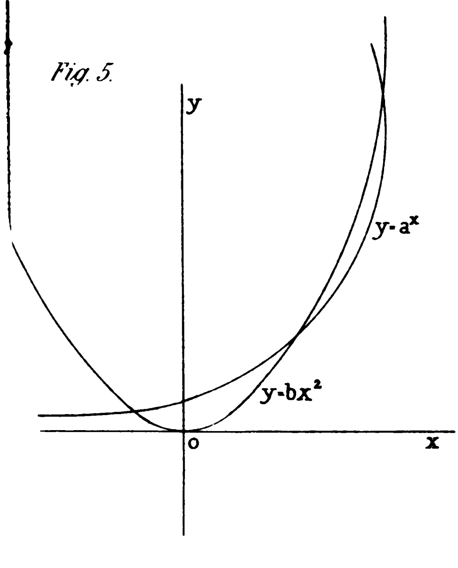

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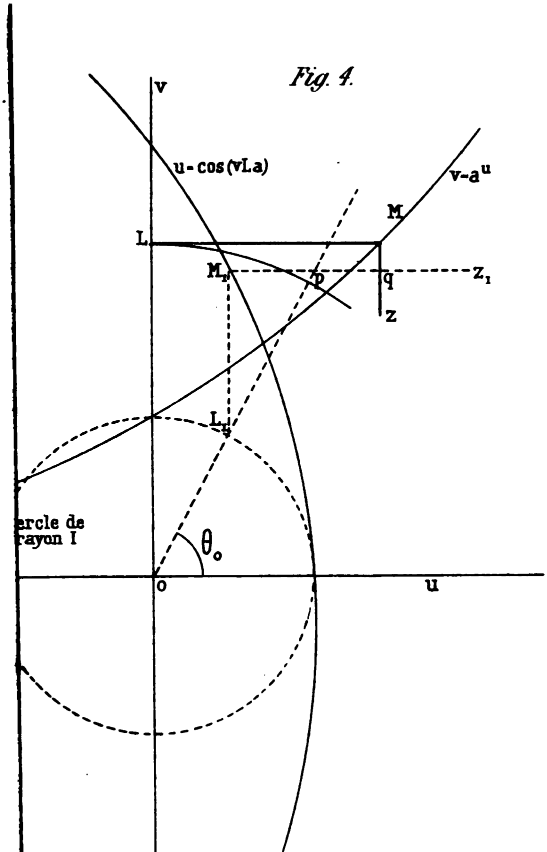

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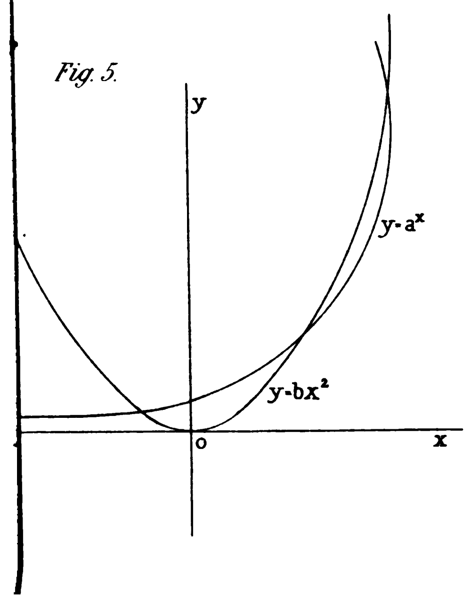

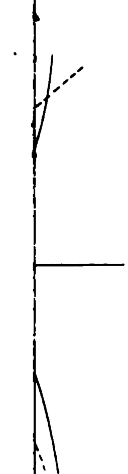

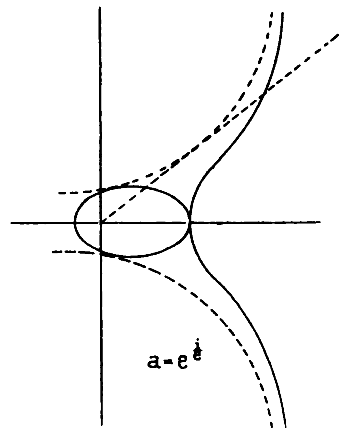

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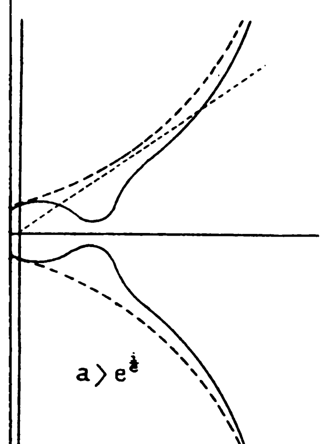

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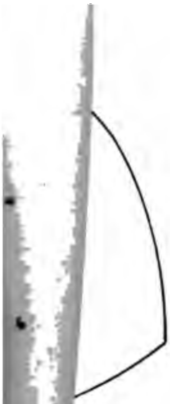

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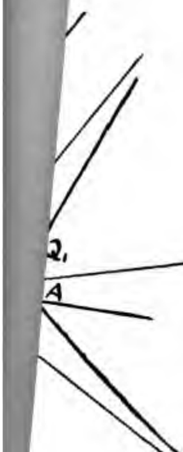

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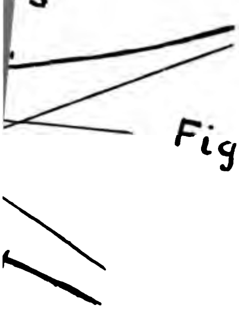

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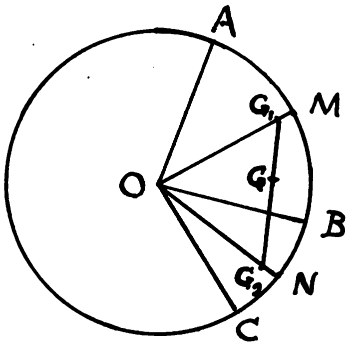

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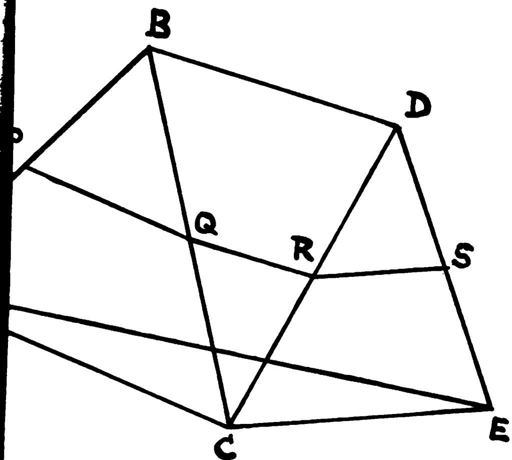

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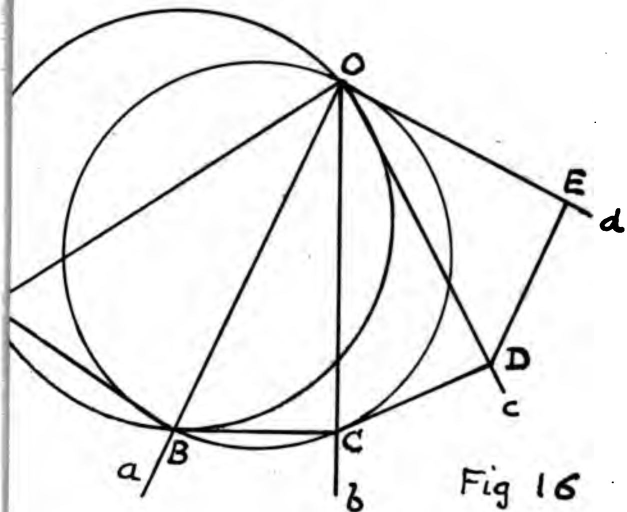

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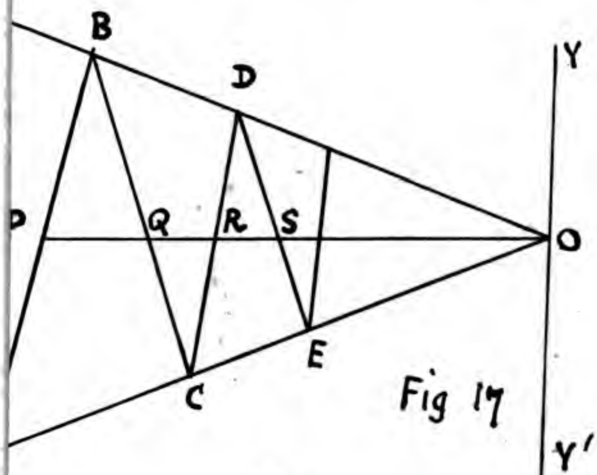

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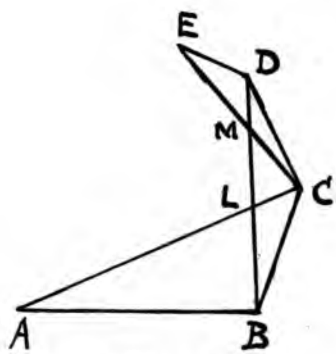

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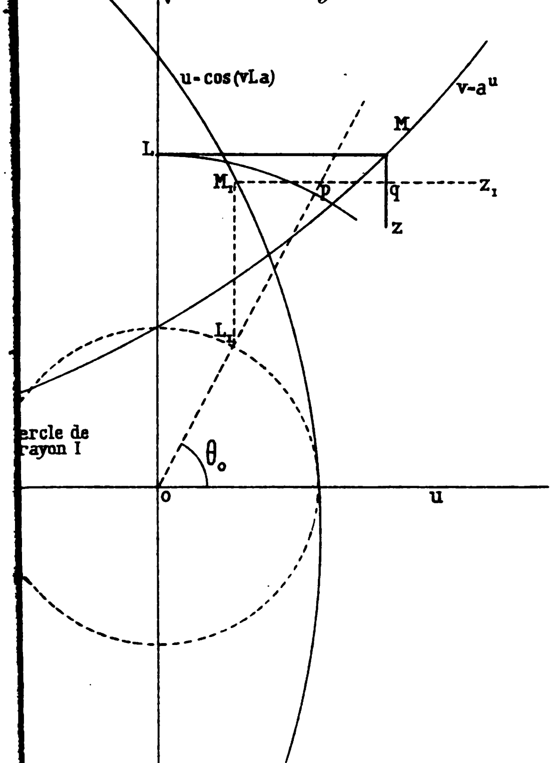

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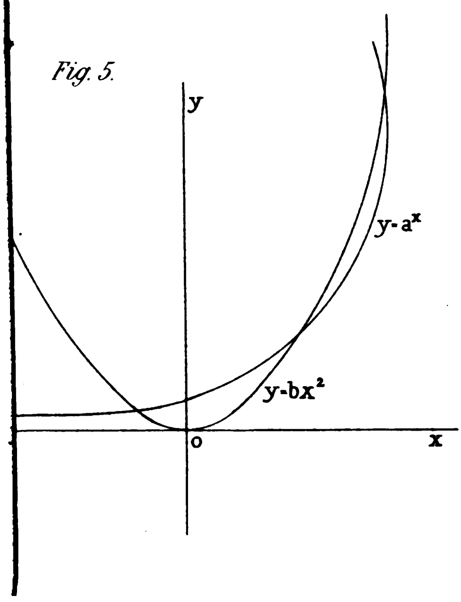

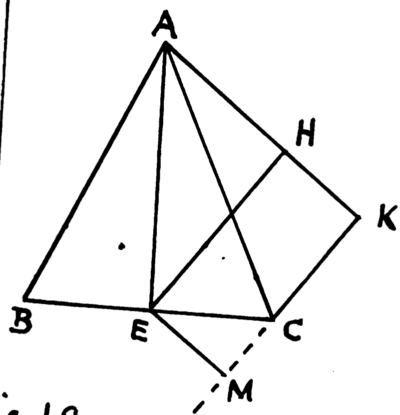

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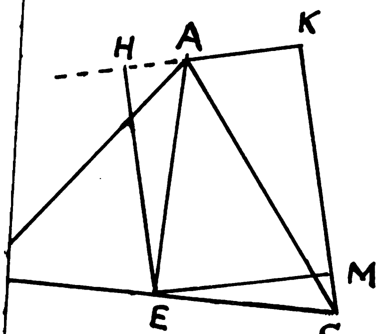

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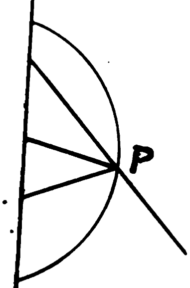

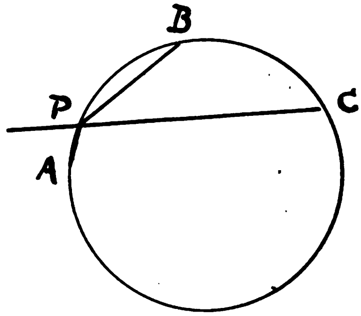

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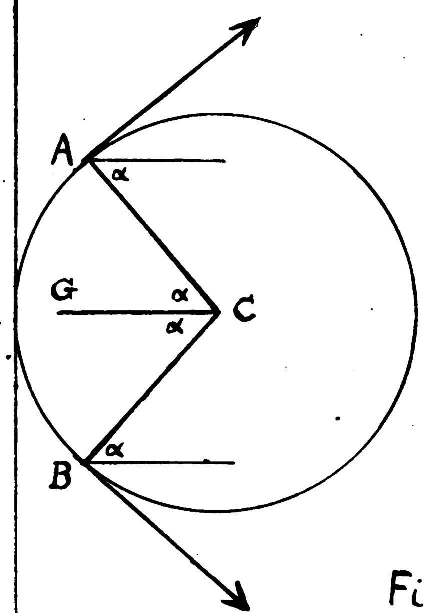

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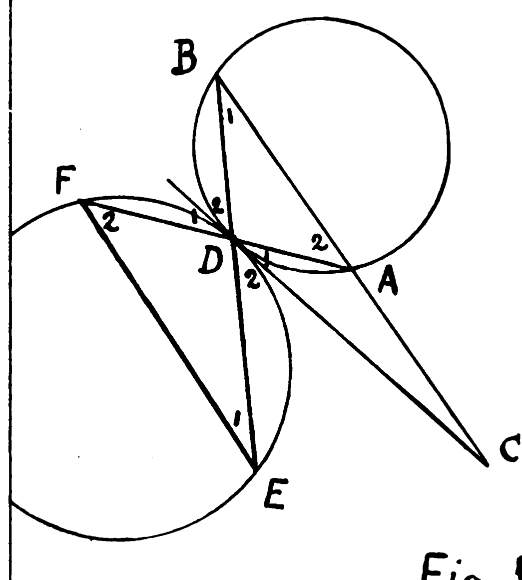

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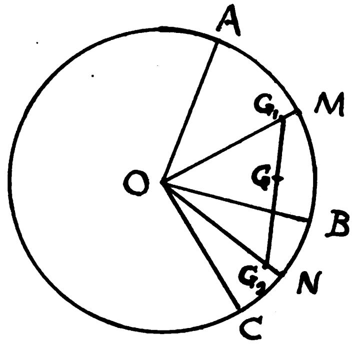

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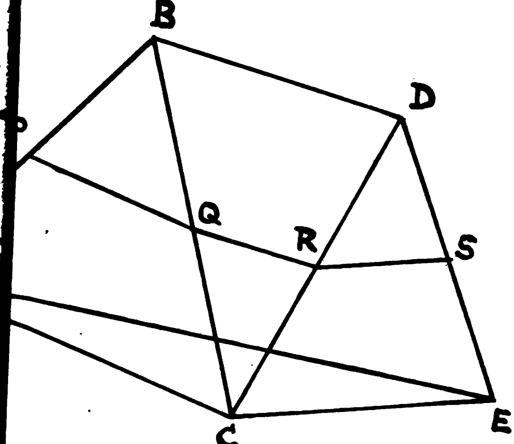

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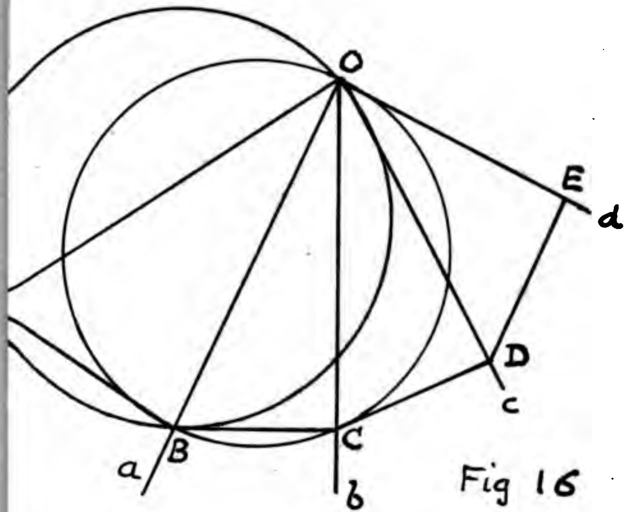

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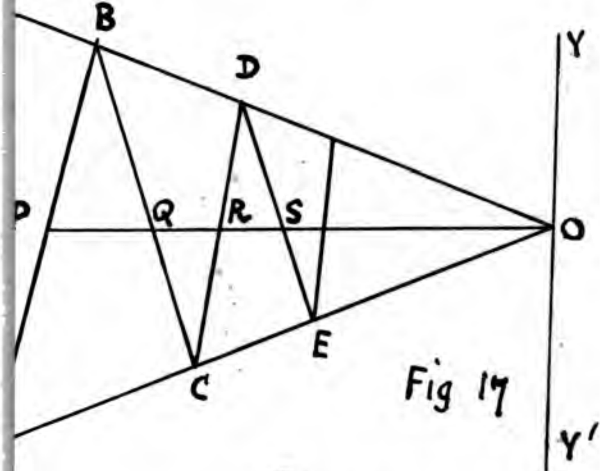

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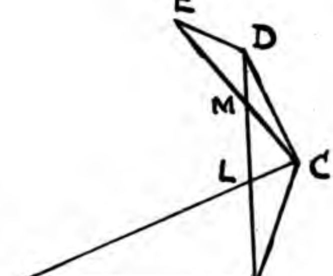

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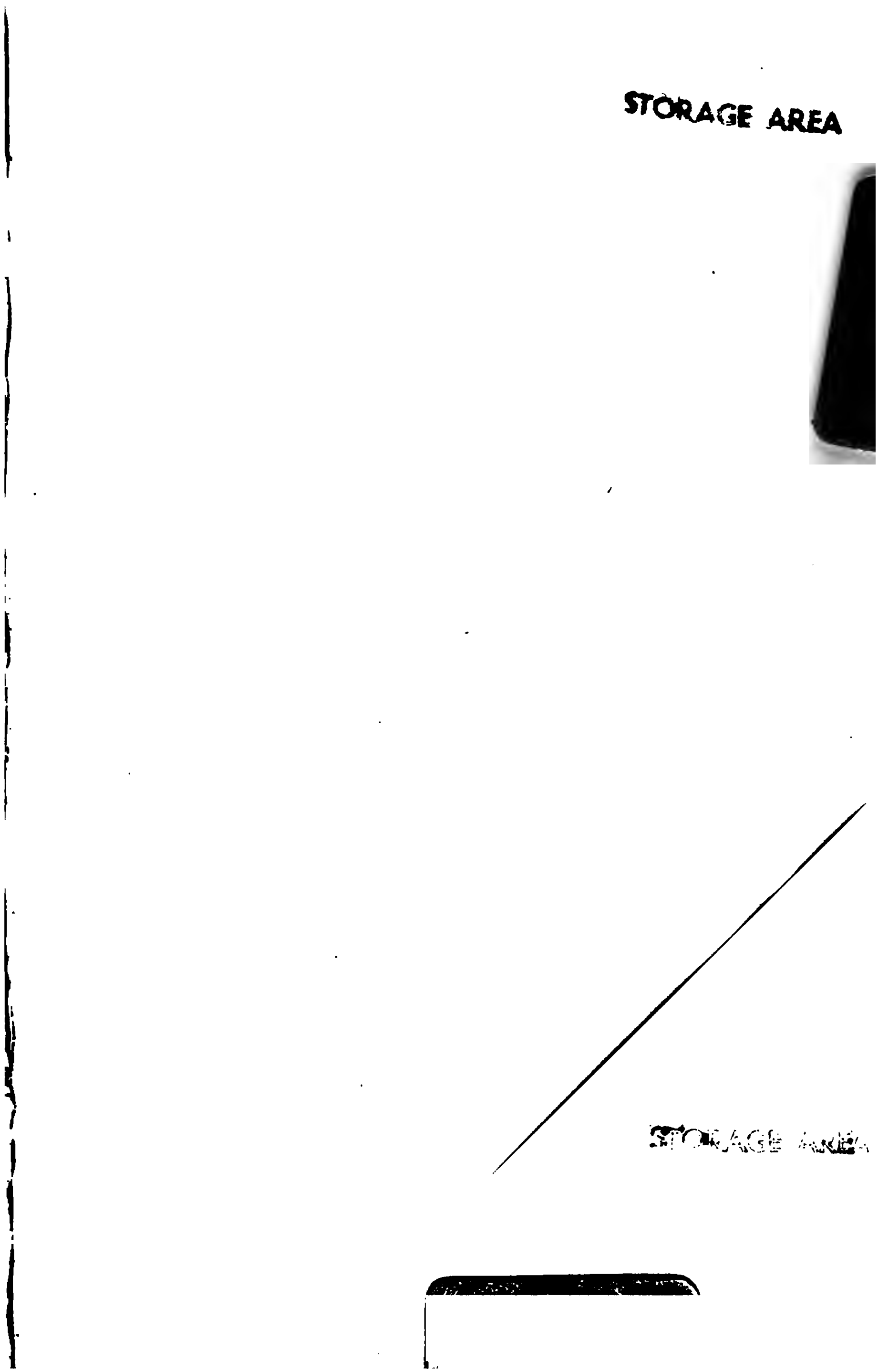
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